Liability Insurance Contracts*

Jorge Lemus[†], Emil Temnyalov[‡], and John L. Turner[§]

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Abstract

We study the market for third-party liability insurance, where a risk-neutral agent buys insurance to cover the legal costs and damages from a liability lawsuit (e.g., patent infringement). Third-party insurance is valuable because it improves the agent's bargaining position when negotiating a settlement, but it may incentivize the agent to litigate rather than settle out of court. In a competitive market, contrary to classical results on first-party insurance, only an inefficient pooling equilibrium with underinsurance may exist. In a monopolistic setting, an insurer offers at most two contracts, which underinsure less risky types and may induce high types to litigate.

JEL Code: D82, G22, K1, K4.

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[†]University of Illinois Urbana-Champaign, Department of Economics. jalemus@illinois.edu

[‡]University Technology Sydney, Business School. emil.temnyalov@uts.edu.au

[§]University of Georgia, Department of Economics. jlturner@uga.edu.

1 Introduction

Third-party liability insurance is fundamentally different from first-party insurance: in the former setting, an agent buys insurance to protect against liability for loss or damage caused to a third party (e.g., patent infringement, product liability, employment-related liability, or malpractice); whereas in the latter setting, the agent buys insurance to protect itself against losses (e.g., health, life, or property insurance). Importantly, in the case of liability insurance, claims for compensation require costly assignment of responsibility between the policy holder and the third party, because a court needs to determine whether the agent is responsible for the loss incurred by the third party. Most such cases are in fact settled out of court, due to the costs involved in the legal process, and liability insurance is valuable because it affects the agent's payoff from negotiating a settlement. Hence this source of value is distinct from the value of first-party insurance, where the agent's loss is generally clear and involves no litigation, and the value of insurance derives from risk aversion.

Most of the theoretical literature on insurance and adverse selection—including the seminal work of Rothschild and Stiglitz (1976) and Stiglitz (1977)—studies first-party insurance. In reality, however, third-party insurance is pervasive and much less studied by economic theory. We fill this gap by modeling third-party insurance and showing new features that are not captured in the classical framework of first party insurance.

We consider a model where an agent buys liability insurance, a third party sues the agent for damages, and then the agent and third party have the option to bargain over a settlement or to continue with litigation; the probability that the agent is liable is generally private information. We study liability insurance with adverse selection and moral hazard under two canonical market structures: in a competitive market for insurance; and in a monopoly provider, where the monopolist designs and prices insurance contracts. In both settings we find that contracts for third-party insurance differ significantly from the first-party insurance contracts considered in the literature. First, in a competitive market, we find that there can only be pooling equilibria, because the classical intuition of "cream skimming" (Rothschild and Stiglitz, 1976) is reversed with third-party insurance. Second, with a single seller, we show that the optimal mechanism consists of menu of at most two contracts—one that covers legal costs only, and one that covers legal costs and partially covers damages payments.

Our results on the equilibrium of a competitive market for third-party liability insurance are in sharp contrast to the seminal work in Rothschild and Stiglitz (1976) where only separating contracts can be offered in equilibrium because of "cream skimming": in a candidate pooling equilibrium, an insurer would be able to profitably deviate by offering a contract that only attracts types who generate positive surplus, which undermines the cross-subsidization necessary for sustaining the pooling equilibrium. In contrast, in the case of third-party insurance that we study, cross-subsidization is not necessary and the cream skimming effect is reversed: here, an insurer can only separately attract types who generate negative payoffs, and therefore has no incentive to undermine the pooling equilibrium. Moreover, this intuition also implies that separating equilibria do not exist. Our findings thus point out that the canonical model of insurance is more appropriately applied to first-party insurance, while third-party liability insurance requires a richer model that also considers the effect of insurance on the agent's ex post actions.

Our results on the optimal mechanism with a single seller also differ from existing results on insurance contracts, such as in Chade and Schlee (2012), where the optimal menu price discriminates among different agent types, because we find that the insurer will offer at most one contract that covers damages. In fact, the insurer's problem of designing a menu of liability insurance contracts is one of mechanism design with a non-differentiable value function, where the non-differentiability arises because the agent has a non-contractible ex-post action—to settle or litigate. This choice introduces a novel type of ex-post moral hazard which does not appear in first-party insurance, because the insurance changes the agent's incentives to settle. This type of moral hazard problem shows up in the seller's mechanism as an additional ex post incentive constraint. We find that in general the insurer wants to fully cover the legal costs of all agent types, and to partially cover the damages payments of a subset of relatively high types (who are the more "risky" types). In some cases, the optimal contract may even induce inefficient litigation in equilibrium, where in the absence of insurance there would have been no litigation. This points to novel potentially negative welfare effects of liability insurance, which have not been explored in the literature.

In general, liability insurance is valuable because it improves the agent's bargaining position when negotiating a settlement with a third party. For example, insurance for patent damages compensates the policy holder only after a patent owner proves in court that in fact the policy holder has infringed on its patents. This costly legal process to establish liability creates the possibility for the policy holder and the third party to bargain in order to settle their dispute and save on the legal costs. Insurance improves the agent's payoff from litigation, and therefore also the payoff from bargaining, so insurance is valuable regardless of the agent's attitude towards risk. Indeed, most types of liability insurance are bought by firms, and in this case it is natural to think of the agent as risk neutral. Intuitively, insurance allows the insurer and the agent to extract surplus from the third party via a more favorable settlement. As the level of damages insurance increases, the agent's payoff from litigation increases, which improves its outside option in bargaining with the third party. Hence third-party insurance is valuable even in a risk neutral environment.

To explain our results in both competition and monopoly, first consider the case of perfect information, where the insurer knows the agent's type, which is the probability of being found liable for damages. Notice that as the insurance becomes more generous, the agent's litigation payoff increases and the surplus from bargaining over a settlement decreases; if insurance is too generous, the bargaining surplus would in fact be negative. The first-best contract increases the agent's payoff from litigation to the point where it is exactly indifferent between litigating or settling. With perfect information, the optimal contract would be individually tailored to each type, and the surplus would be captured by either the insurer or the agent, depending on whether the market is monopolistic or competitive.

With asymmetric information, however, the equilibrium changes dramatically, and in ways quite different from markets for first-party insurance. First, with asymmetric information and perfect competition, a pooling equilibrium may exist. Key to this result is that in our setting pooling agents into a single zero-profit contract does not necessarily require cross-subsidization of types. In the equilibrium we characterize, the insurance offered is sold at a price of zero, is optimal for the highest-risk types, but is also bought by lower-risk types. No type litigates in equilibrium, so the insurer never bears any costs. All types gain an improved bargaining position (and a lower settlement payment) in equilibrium. In contrast to the setting of Rothschild and Stiglitz

(1976), it is not possible for another insurer to cream-skim. Because of a lack of cross-subsidization, there is no way such an insurer can target one type and undercut the equilibrium (zero) price.[1]

Second, a separating equilibrium *never* exists in our setting. Intuitively, an insurer always loses money if all contracts induce litigation, so two litigation-inducing contracts are not an equilibrium. Types do not separate if no contracts induce litigation and both are priced to break even. And if there is one contract that induces litigation and one that does not, then the one that induces litigation is a money loser. Hence, the contract that does not induce litigation must have a positive price, so an alternative insurer can cream-skim away the customers that buy the positively-priced contract.

Third, with asymmetric information in a monopolistic mechanism design setting, we find that the optimal mechanism is generally a menu of only two contracts: one that only covers legal costs, designed for low (less risky) types, and one that covers legal costs and partially covers damages, designed for high (more risky) types. The seller wants to cover legal costs for all types, because they are independent of the agent's type. Hence, rather than excluding low types as in the standard mechanism design setting, here the seller can offer such low types a contract which does not introduce any additional information rents for high types. Under the usual regularity condition on the distribution of types, types with negative virtual valuation buy the contract that only covers legal costs, while high types buy a contract that also covers damages. The latter contract is more valuable to all types, but also may induce higher types to litigate rather than settle, if their bargaining surplus becomes negative, which is very costly to the insurer. We show that a certain form of monotonicity holds: if in the optimal mechanism a type chooses to litigate, all higher types must also litigate; moreover, the set of types who are exactly indifferent is of measure 0. Taken together, these two facts

[1] In certain circumstances it may be possible to sell profitably an alternative pooling contract, which destroys equilibrium. For example, with just two types, a pooling equilibrium fails to hold if the fraction of high-risk types is very low. Intuitively, any increases in the generosity of insurance (away from the equilibrium contract) induce litigation by high-risk types. Any profitable alternative contract must both be more attractive to low-risk and high-risk types, and be priced high enough to overcome the losses due to litigation by high-risk types. When the fraction of high-risk types is sufficiently high, it is impossible to do this. This characteristic echoes Rothschild and Stiglitz (1976), where the presence of too few high-risk types upsets the candidate separating equilibrium.

allow us to characterize the optimal mechanism despite the non-differentiability of the value function. The seller's problem with respect to damages coverage can be re-cast as one of choosing a litigation cutoff type of the agent, where higher-risk types litigate and lower-risk types settle (while even lower types buy the contract that excludes damages altogether). Indeed, we find that under some conditions the optimal menu of contracts induces some high types to litigate in equilibrium, while in the absence of insurance all types would settle.

In what follows, we first discuss in more depth one of our main motivating examples of liability insurance: patent infringement. In Section 2 we review some of the relevant literature. Section 3 lays out the model and presents our results in the cases of competition and monopoly. Section 4 discusses the main findings and proposes several directions for future research.

Motivating Example of Liability Insurance: Patent Litigation

The amount of patent litigation in the United States has grown rapidly after the establishment of the Court of Appeals for the Federal Circuit, in 1982, and steeply increased after 2004 (Bessen et al., 2015; Tucker, 2016). This more recent surge—which increased the number of cases from about 2,500 to 5,000 per year—was largely been driven by litigation initiated by patent assertion entities ("PAEs"), also called "patent trolls," who in recent years have filed more than half of all patent lawsuits in the U.S.[2]

Patent litigation is costly for firms and entrepreneurs (Bessen et al., 2011). Although markets for patent litigation insurance have existed in the United States since at least the 1980s, the recent increase in patent litigation has spurred growth and activity in the market for insurance. Firms such as RPX Corporation, IPISC, Triology, and InsureCast now offer insurance to entrepreneurs and firms to cover some fraction of the legal costs or damages that they may have to pay as defendants in an infringement lawsuit.[3] These companies offer both offensive and defensive insurance contracts. The former is used

[2]See, for example, Chien (2009) and Tucker (2016).

[3]For other companies offering Patent Infringement Insurance, please visit: http://wspla.org/wp-content/uploads/2016/09/Appendix-Insurance-Coverage-for-Patent-Infringement-Lawsuits.pdf

by patent owners to pay for the cost of enforcing their patents, while the latter is used by producing firms accused of patent infringement to cover the legal costs and penalties imposed by a court following a lawsuit.[4] A particular feature of these contracts is the freedom of the policy holder to decide whether to settle or to litigate. Consider, for example, the description of the policy offered by Triology Insurance:[5]

"The Policy Holder controls the lawsuit. The Company may suggest reliable and preferred counsel to the Insured but the Insured ultimately chooses [...] The Insured dictates the settlement terms, if any, not the Company."

In Europe, the market for patent insurance has also been active.[6] A study in 2006 for the European Commission proposed to make patent insurance mandatory for small-to-medium-sized enterprises.[7] Fuentes et al. (2009) describes the economic trade-offs behind this proposal.

2 Literature Review

There is an extensive literature on insurance in law (Schwarcz and Siegelman, 2015) as well as in economics (Dionne, 2013). The problem of first party insurance contracts has been heavily studied, while third party (or liability) contracts have not been fully explored. In particular, under adverse selection and ex-post moral hazard research on liability insurance contracts is scarce, while first party insurance markets have been analyzed by Rothschild and Stiglitz (1976) (perfect competition) and by Stiglitz (1977) and Chade and Schlee (2012) (monopolist). Our framework includes a bargaining game whose outcome depends on the private information of the agent and the insurance contract purchased. Since we focus on risk-neutral agents, we can use mechanism design to compute the optimal monopoly contract. However, given the nature of the cost of the insurer, our model also relates to the work of Carbajal and Ely (2013) on

[4] To see specific details on some of the contracts, visit the following links:
http://www.patentinsuranceonline.com/defense/index.html
https://www.rpxcorp.com/rpx-services/rpx-patent-litigation-insurance/

[5] http://www.trilogyinsurancegroup.com/services/defense-insurance

[6] http://jolt.law.harvard.edu/digest/patent/insuring-patents

[7] http://ec.europa.eu/internal_market/indprop/docs/patent/studies/pli_report_en.pdf

optimal mechanisms with non-differentiable value functions.

Since the action taken by the agent—to settle or to litigate—after insurance has been bought affects the payoff of the insurer (ex-post moral hazard), our paper also relates to the literature on optimal contracting under adverse selection and moral hazard (Picard, 1987; Guesnerie et al., 1989). For example, in the context of health insurance, Albert Ma and Riordan (2002) study the demand for treatment under managed care. Chiappori et al. (2006) shows that coverage and ex-post risk are positively correlated and Finkelstein and Poterba (2004) find systematic relations between ex post mortality and annuity characteristics, which is consistent with adverse selection. The key driving force in our model is the improved bargaining position of an insured agent. Several papers, including Kirstein (2000), Van Velthoven and van Wijck (2001), Kirstein and Rickman (2004), and Llobet et al. (2012) have shown how insurance may have value for risk-neutral buyers, by making litigation credible or improving the policy holder's bargaining position. However, none of these papers study equilibrium under adverse selection, so their results are qualitatively very different from ours.

Specifically on liability insurance, Shavell (1982) studies the effect of liability insurance on ex-ante moral hazard (demand for care) in a model without bargaining, while we focus on ex-post moral hazard and bargaining given the equilibrium contracts under different market structures. Meurer (1992) studies why it may be optimal for the insurer to offer a contract where it controls the litigation and settlement process on behalf of the insured, while limiting the coverage of damages up to some cap, which creates a conflict of interest. In contrast, we focus on the case where the insured controls the litigation and settlement process.

While to the best of our knowledge our paper is the first one to present a framework to analyze defensive patent insurance under adverse selection, there is some work on offensive patent insurance, where an insurer funds the agent's lawsuit against a third party. This work is clearly related, but an entirely different type of model. Llobet et al. (2012) discuss policies that cover a fraction of litigation costs incurred in enforcing a patent. Insurance makes it less costly to enforce patents and improves the patent holder's bargaining position, allowing it to sometimes deter entry by a potential entrant. Buzzacchi and Scellato (2008) similarly focuses on offensive insurance. Duchene (2015) also studies offensive insurance and shows that with private information, patentees have some incentive to obtain insurance for its commitment value, but may opt not to buy it because of an inability to sharply signal and avoid pooling equilibria. In contrast to this literature, defensive insurance introduces some new issues: first, there is no gain to the insuring party from making litigation threats credible; second, when insurance covers damages, the insurer is exposed to significant losses when litigation occurs. Our analysis shows that both of these factors matter for the insurer's optimal contract.

Historically, markets for third-party insurance have shown more volatility than first-party insurance markets. In 1986 in the United States, for example, premiums rose sharply and some insurers declined to sell certain types of coverage. In the wake of this crisis, a number of papers sought to explain how liability insurance differs from other insurance (Priest, 1987; Winter, 1991; Harrington and Danzon, 1994). This literature analyzes the difficulties insurers have in forecasting liability losses, but does not focus on the role of the value of insurance under bargaining.

Our work also relates to the literature on contingent fees. Under such contracts, lawyer fees are reduced but the lawyer keeps part of any payments awarded. Hence, similar to our model, the insured party may not face the full consequences of a litigation outcome. Dana and Spier (1993) show that contingency fees help solve an agency problem. Intuitively, an attorney who is paid using a contingency fee has better incentives to provide accurate information to her client about the strength of the case. Rubinfeld and Scotchmer (1993) study a Rothschild-Stiglitz-style competition model of this service. They make the related points that clients with high-quality cases can signal their cases' strength by preferring hourly fees, while attorneys can signal their ability by preferring contingency fees. Similar points have been made by Gravelle and Waterson (1993).

Finally, a number of papers review the large literature on litigation and settlement, including Hay and Spier (1998) and Spier (2007). Reinganum and Wilde (1986) present a model of settlement and litigation under different rules for cost allocation: the loser pays rule and the case where each party pays their own costs.

3 Model

Consider a risk-neutral agent (A) selling a product or providing a service. The agent may cause damage or harm to a third party (TP) giving rise to a legal liability. The agent may purchase third-party liability insurance from a risk neutral insurer (I) to cover the legal costs and damages if the court determines that the harm in fact has occurred. Going to trial is costly for both parties: A pays a cost $c_A > 0$ and TP pays $c > 0$. If the court determines that the agent is liable, the agent needs to make a payment d to the third party. Figure 1 describes the timing of the model.

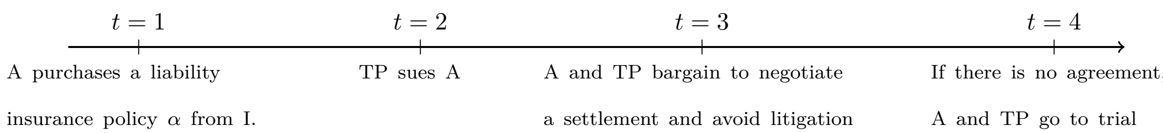


Figure 1: Timing of the events in the model.

At $t = 1$, the agent considers buying liability insurance. Insurance contracts are defined by $\alpha = (\widehat{\alpha}_L, \widehat{\alpha}_D)$, meaning that the insurer will pay $\widehat{\alpha}_L$ to cover the litigation costs and $\widehat{\alpha}_D$ to cover damages.[8] The set of contracts that the insurer can offer is

$$\mathcal{A} = \{(\widehat{\alpha}_L, \widehat{\alpha}_D) \ : \ \widehat{\alpha}_L \in [0, c_A], \ \widehat{\alpha}_D \in [0, d]\}.$$

We consider different market structures and different levels of information in the market for liability insurance. As in Rothschild and Stiglitz (1976), we analyze the case of a perfectly competitive insurance market and agents privately informed about the probability of liability. As in Stiglitz (1977) and Chade and Schlee (2012), we also analyze the case of a monopolist insurer which is either informed of not informed about the agent's liability risk.

After a lawsuit is filed, at $t = 2$, the agent and the third party can assess the probability of liability p, so they bargain at $t = 3$ under complete information.[9] The agent's bargaining payoff at $t = 3$ depends on the probability of liability p, the insurance contract it has bought, and the decision to settle or to go to trial. The third party does

[8] In Appendix B.2 we allow for insurance policies to also cover settlements. We show that in the optimal contract that insurer does not cover settlements, so the main results of our paper are unaffected.

[9] We later also discuss the case of bargaining under incomplete information.

not have a credible litigation threat if $pd \leq c$. For a given agent's type p and contract $\alpha \in \mathcal{A}$, the agent's expected payoff from going to litigation is

$$V(p, \alpha, L) = \begin{cases} -(c_A - \widehat{\alpha}_L) - p(d - \widehat{\alpha}_D) & \text{if } pd \geq c, \\ 0 & \text{if } pd < c. \end{cases} \tag{1}$$

Notice the importance of the litigation costs in Equation 1: if $c_A = c = 0$, this is precisely the Rothschild and Stiglitz (1976) framework under risk neutrality.[10]

Given the need for a legal proceeding to determine liability, it is natural to permit the agent and the third party the option to settle the dispute out of court. At $t = 3$, the agent and the third party Nash-bargain over a fee to settle the lawsuit. The agent's bargaining power is $\theta \in [0, 1]$. Since bargaining occurs under complete information, settlement occurs with probability one if and only if the bargaining surplus is non-negative. Notably, when the agent does not have insurance, it is *always* optimal to settle litigation out of court, because the bargaining surplus is $c_A + c > 0$.

We now consider the effect of the insurance contract on the decision to settle or not. Notice that the bargaining surplus under insurance policy $\alpha = (\widehat{\alpha}_L, \widehat{\alpha}_D)$ is

$$S_B = c + c_A - \widehat{\alpha}_L - p\widehat{\alpha}_D.$$

It is important to distinguish two cases: $\widehat{\alpha}_D = 0$ and $\widehat{\alpha}_D > 0$. If $\widehat{\alpha}_D = 0$, this surplus is positive and independent of the liability probability, so there is always settlement. However, since the settlement fee is proportional to the joint surplus, the agent is able to pay a lower settlement fee, since having insurance improves its bargaining position. Within the smaller class of contracts with $\widehat{\alpha}_D = 0$, the contract that maximizes the value of insurance for the agent is $\widehat{\alpha}_L = c_A$.

If $\widehat{\alpha}_D > 0$, the bargaining surplus depends on the probability of liability and could be negative, meaning that a settlement agreement reduces the bargaining surplus, so the

[10] In the setting of Rothschild and Stiglitz (1976), an individual has initial wealth of W, which is 0 without loss of generality in our case since we have risk neutrality. The individual will suffer the loss of d with probability p. This individual can purchase α_1 units of insurance which pay $\widehat{\alpha}_2$ if the loss occurs. Let $\alpha_2 \equiv \widehat{\alpha}_2 - \alpha_1$, and let $\alpha \equiv (\alpha_1, \alpha_2)$ fully specify the insurance contract. Conditional on the insurance bought, the individual's expected utility is $V(p, \alpha) = (1 - p)U(W - \alpha_1) + pU(W - d + \alpha_2)$. Under risk neutrality and no costs, $V(p, \alpha) = W - \alpha_1 - p(d - \hat{\alpha}_2)$.

parties will not agree, and will go to trial. Thus, insurance contracts with $\widehat{\alpha}_D > 0$ may induce litigation, in contrast with the case where the agent does not have insurance and always settles. The bargaining surplus may be negative for high risk types, i.e., agents with a larger probability of liability

$$p > p^* \equiv \frac{c + c_A - \widehat{\alpha}_L}{\widehat{\alpha}_D}. \tag{2}$$

When $\frac{c}{d} \leq p \leq p^*$, reaching a settlement agreement increases the joint surplus, and the agent's payoff is then

$$\begin{aligned} V(p, \alpha, S) &= -(c_A - \hat{\alpha}_L) - p(d - \hat{\alpha}_D) + \theta S_B \\ &= V(p, 0, S) + (1 - \theta)(\widehat{\alpha}_L + p\widehat{\alpha}_L) \end{aligned}$$

When $p > p^*$, reaching a settlement agreement lowers the bargaining surplus, so litigation becomes unavoidable. In this case, the agent's payoff is given by

$$V(p, \alpha, L) = -c_A - pd + \widehat{\alpha}_L + p\widehat{\alpha}_D.$$

Insurance increases the payoff of a low-risk agent by improving its bargaining position. The agent only captures a fraction $(1-\theta)$ of the savings induced by a better bargaining position. Notice the insurance does not provide any value for an agent that has all the bargaining power and settles, since the agent was already capturing all the bargaining surplus. Insurance allows an agent that settles to capture more of the bargaining surplus. High-risk agents will go to trial and part of their expenses will be covered by the insurer. Figure 8 describes the effects of insurance on the decision to reach a settlement agreement.[11]

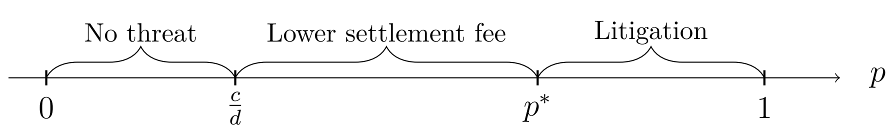


Figure 2: The effect of insurance on litigation for different types of agents.

Lemma 1. *Consider an insurance policy $\alpha = (\widehat{\alpha}_L, \widehat{\alpha}_D) \in \mathcal{A}$ and p^* as defined in (2). The agent's willingness to pay for insurance, $W(p, \alpha)$, and the expected cost for the*

[11]The agent faces no threat for $p < \frac{c}{d}$. For the remainder of the paper we restrict attention to $p \geq \frac{c}{d}$.

insurer of providing policy α to an agent of type p, $K(p,\alpha)$, are given by

$$W(p,\alpha) = \begin{cases}(1-\theta)(c+c_A)+(p-p^*)\widehat{\alpha}_D(1-\theta) & \text{if } p \le p^* \\ (1-\theta)(c+c_A)+(p-p^*)\widehat{\alpha}_D & \text{if } p > p^*\end{cases}, \tag{3}$$

$$K(p,\alpha) = \begin{cases}0 & \text{if } p \le p^* \\ c+c_A+(p-p^*)\widehat{\alpha}_D & \text{if } p > p^*\end{cases}. \tag{4}$$

Proof. Directly from the definitions. See the Appendix for details. □

Equation (3) shows that the willingness to pay is a continuous and convex function of p with a kink at p^*. Also, it depends on $\widehat{\alpha}_L$ implicitly through the definition of p^*. From equations (3) and (4) it is easy to see that the willingness to pay for insurance is always lower than the cost of providing it for high risk types that choose to litigate, i.e., for types $p > p^*$. In fact, the difference between the willingness to pay and cost is exactly $\theta(c + c_A)$ for $p > p^*$. Figure 3 depicts the willingness to pay and the cost of providing an insurance contract α to an agent of type p.

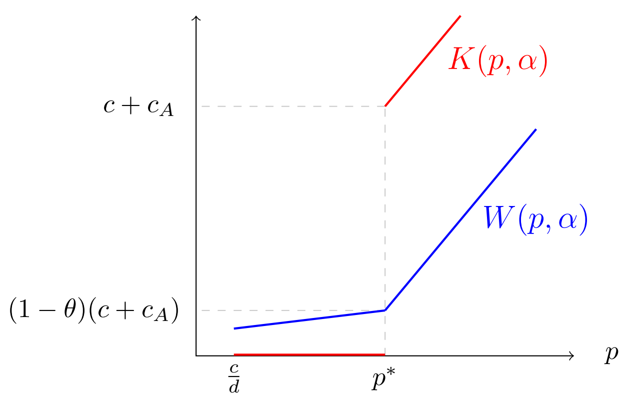


Figure 3: $W(p,\alpha)$ is the entrepreneur of type p's willingness to pay for insurance policy α. The cost for the insurer of providing the coverage prescribed by policy α for an entrepreneur of type p is given by $K(p,\alpha)$. The type that is indifferent between settlement and litigation is p^*.

Corollary 1. *We have:*

1. *The willingness to pay for contract* $(\widehat{\alpha}_L, \widehat{\alpha}_D) = (c_A, 0)$ *is* $(1-\theta)c_A$.

2. For any $p > p^*$ and for any policy α we have $K(p,\alpha) - W(p,\alpha) = \theta(c + c_A)$.

A contract that fully covers litigation costs but not damages always induces settlement. An agent protected by an insurance policy that fully covers litigation costs is equivalent (from the third party's perspective) to an agent with litigation costs equal to zero. Thus when facing an agent with full litigation costs coverage, the plaintiff is unable to capture the surplus $(1-\theta)c_A$ that it otherwise would capture. This reduction in the bargaining surplus reduces the settlement fee the agent pays the third party. This amount is what the agent is willing to pay for the insurance policy that fully covers litigation costs. The second part of Corollary 1, shows that when the agent litigates instead of settling, the joint surplus of the insurer and the agent decreases by $\theta(c+c_A)$, which is the bargaining surplus that would otherwise be captured by the agent in the settlement negotiation.

Next, we can show that although insurance contracts are generally characterized by two parameters, we can reduce the set of contracts that we must consider, since some contracts are weakly dominated from the insurer's perspective.

Lemma 2. *An insurance contract $\alpha = (\widehat{\alpha}_L, \widehat{\alpha}_D)$ with $\widehat{\alpha}_L < c_A$ is weakly dominated by an alternative contract with $\hat{\alpha}_L = c_A$.*

Proof. Consider a contract $\alpha = (\widehat{\alpha}_L, \widehat{\alpha}_D)$, with $\widehat{\alpha}_L < c_A$. Let $p^* \equiv p^*(\alpha)$ the type that is indifferent between settlement and litigation. Consider a contract $\alpha' = (c_A, \widehat{\alpha}'_D)$ such that $p^*(\alpha') = p^*$. This contract leaves the same type p^* indifferent and $\widehat{\alpha}'_D < \widehat{\alpha}_D$. In this new contract, the willingness to pay for types $p < p^*$ $(p > p^*)$increases (decreases). By Corollary 1, the difference between cost and willingness to pay is constant, regardless of the contract, $K(p,\alpha) - W(p,\alpha) = \theta(c + c_A)$. Even more, if the agent has private information regarding p, the reduction in the willingness to pay for high-risk types under the contract α' implies that fewer types $p > p^*$ are willing to buy insurance, for a given price, compared to the original contract α. This is good for the insurer since it avoids losses. Therefore, α' weakly dominates α from the perspective of the insurer. □

By Lemma 2, the space of contracts can be characterized by the single parameter p^*, representing the contract $(\hat{\alpha}_L, \hat{\alpha}_D) = \left(c_A, \frac{c}{p^*}\right)$. We allow for $p^* = +\infty$, representing the contract $(c_A, 0)$ that fully covers litigation costs but does not cover damages. Hence, from now on, we can identify the space of contracts with $p^* \in \left[\frac{c}{d}, \infty\right]$. We re-write equations (3) and (4), the value of a contract p^* to an agent of type p, and the insurer's cost of providing a contract p^* for an agent of type p, using the single parameter p^* to characterize different contracts,

$$W(p,p^*) = \begin{cases} (1-\theta)\left[c_A + c\dfrac{p}{p^*}\right] & \text{if } p \leq p^* \\ \left[c_A + c\dfrac{p}{p^*}\right] - \theta(c + c_A) & \text{if } p > p^* \end{cases}, \tag{5}$$

$$K(p,p^*) = \begin{cases} 0 & \text{if } p \leq p^* \\ c_A + c\dfrac{p}{p^*} & \text{if } p > p^* \end{cases}. \tag{6}$$

Equation (5) shows that the willingness to pay has a kink at p^*. Agents of type $p \leq p^*$ will settle and they will be able to capture an extra $(1-\theta)\left[c_A + c\dfrac{p}{p^*}\right]$ bargaining surplus in the negotiation from their improved bargaining position. Types $p > p^*$ will litigate. By going to litigation, they save $\left[c_A + c\dfrac{p}{p^*}\right]$ on costs, paid by the insurer, but they also lose the bargaining surplus they used to collect from the settlement negotiation. Equation (6) shows the cost of serving type p with contract p^*. When the agent settles, i.e., $p \leq p^*$, the insurer does not incur costs. However, when the agent goes to litigation, the insurer covers its legal costs and some part of the damages.

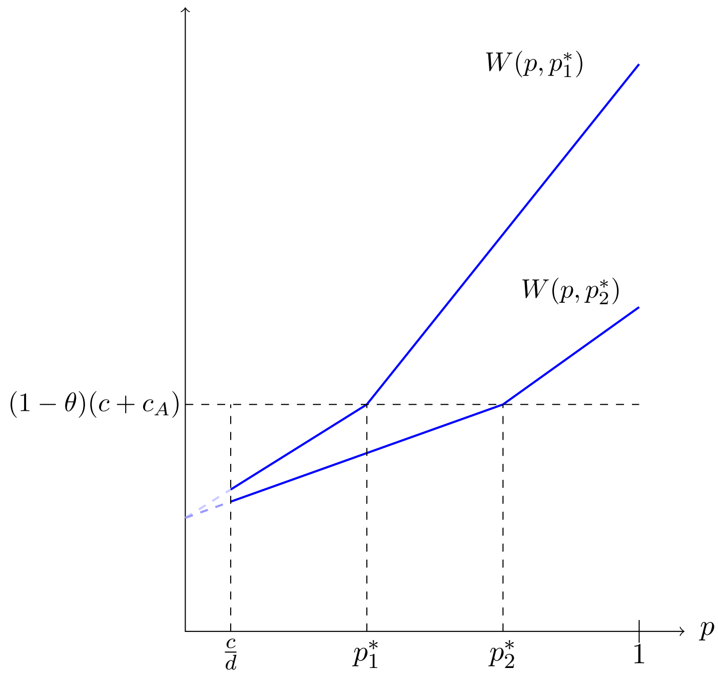


Figure 4: Willingness to pay for two insurance policy contracts indexed by p_1^* and p_2^*.

An important feature of the willingness to pay for insurance is that high-risk types always prefer policies that leave lower-risks types indifferent. Figure 4 shows two policy contracts p_1^* and p_2^* with $p_2^* > p_1^*$. We can see from the figure that for any type p we have that $W(p, p_1^*) > W(p, p_2^*)$ and that $W(p, p_1^*) - W(p, p_2^*)$ is increasing in p.

Corollary 2. *Let $\tilde{W}(p, p^*) = W(p, 1-p^*)$. Then, $\tilde{W}(p, p^*)$ is supermodular.*

3.1 Complete Information

As a benchmark, we present the case of complete information and the case of no adverse selection. With complete information, the insurer sells the contract that most improves the bargaining position of the agent without inducing litigation. This is optimal in the case of competition or monopoly, although the price of the policy is different in the two cases.

Lemma 3. *For a monopoly or under perfect competition, if the insurer(s) can observe p, the optimal insurance policy is $\alpha^*(p) = \left(c_A, \frac{c}{p}\right)$. That is, the contract $p = p^*$ to fully* *covers the litigation expenses and covers only part of the damages.*

Proof. Since p is observable and the insurer incur loses by selling a policy that induces litigation, the optimal contract must induce the agent to reach a settlement agreement. Since under settlement there are not costs paid by the insurer, either a monopolist or a competitive market would offer the contract that maximizes the agent's willingness to pay. The monopolist extracts all the surplus and sells it at price $(1-\theta)(c+c_A)$, while a competitive market offers this policy for free. □

Under complete information, the insurance contract reduces the bargaining surplus under settlement. By taking the agent's incentive to litigate to the absolute brink, with damages insurance $\widehat{\alpha}_D(p)$, the optimal insurance contract extract all the bargaining surplus from the plaintiff. Thus, effectively, insurance under complete information transfers rents from the third party to the insurer (in the case of monopoly) or the agent (in the case of perfect competition).

Third Party insurance contracts (TPIC) and First Party insurance contracts (FPIC) have significant differences. First, FPIC have no value for risk-neutral individuals since all value comes from reduction of risk. In TPIC, in contrast, insurance is valuable for risk-neutral individuals since there is costly verification of the harm. This verification gives rise to settlement negotiations, and insurance adds value within that framework, as long as the third party has some bargaining power.

3.2 Symmetric Information (No Adverse Selection)

Consider now the problem of selling insurance without adverse selection, that is, the case of an agent that is uninformed about its type when purchasing insurance. In this instance, every agent is ex-ante identical, so the insurance policies are equally attractive to all of them. Since there is no externalities among agents, there is no reason to offer more than a single insurance policy.

Suppose the insurance policy p^* is sold at price P_M. The willingness to pay of an agent for this contract is $\mathbf{E}_p[W(p,p^*)]$. A monopolist sets $P_M = \mathbf{E}_p[W(p,\hat{p})]$ and extracts all the ex-ante value of uninformed agents. Hence, the profit maximizing contract for the monopolist is given by:

$$p^* \in \arg\max_{\frac{c}{d} \leq \hat{p} \leq \infty} \mathbf{E}_p[W(p,\hat{p}) - K(p,\hat{p})] \tag{7}$$

In a perfectly competitive market, if insurance contract $\hat{p}$ is offered in equilibrium firms must break even, so the price of that contract must be $P_C(\hat{p}) = \mathbf{E}_p[K(p,\hat{p})]$. Agents buy this contract as long as $\mathbf{E}_p[W(p,\hat{p})] \geq P_C(\hat{p})$. Thus, the only contract that is offered in equilibrium must be p^*, the solution to Problem (7).

A perfectly competitive market and the monopolist sell the same contract at different prices. The following proposition characterizes the contract that is offered to an agent that is uninformed about its type at the moment of buying insurance.

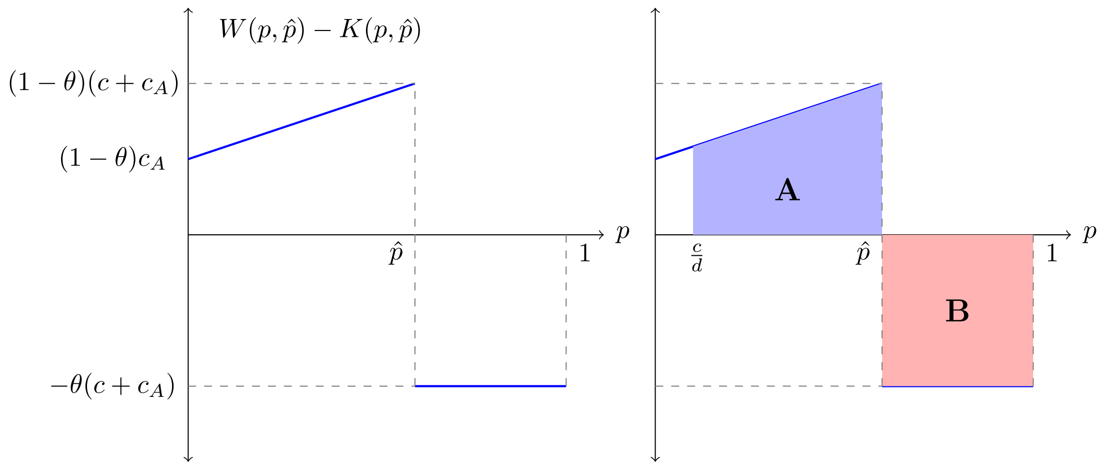
Proposition 1. *With symmetric information and no adverse selection, the insurance policy offered by a monopolist or a perfectly competitive market is characterized by the solution:*[12]

$$\max_{\hat{p} \in \left[\frac{c}{d}, \infty\right]} \quad (1-\theta) \int_{c/d}^{\hat{p}} \left[c_A + \frac{cp}{\hat{p}}\right] dF(p) - \theta(c + c_A)[1 - F(\hat{p})]. \tag{8}$$

Equation 8 in Proposition 1 shows that the optimal contract balances out two kinds of inefficiencies, which are depicted in Figure 5. For a given contract $\hat{p}$, agents with a realized type $p \leq \hat{p}$ will pay a settlement fee that does not capture all the bargaining surplus. Although the bargaining position of the agent improves relative the case of no insurance, it does not improve to the efficient level achieved under perfect information. For this reason, the agent settles for an inefficient settlement fee captured by the first term in Equation (8). This is represented by area A in Figure 5(b). On the other hand, agents that bought insurance and draw type $p > \hat{p}$ will litigate. In this case, the fraction of bargaining surplus $\theta(c + c_A)$ the agent would be able to capture in a settlement negotiation is now lost, represented by area B in Figure 5(b).

Consider the case $\theta = 0$ and a distribution with a continuous density. In this case, the agent does not have any bargaining power in the negotiation with the third party.

[12] In the maximization problem (8) we allow for $\hat{p} = +\infty$, which corresponds to the contract that does not cover damages.



(a) Joint Surplus

(b) Monopolist trade-off

Figure 5: Panel (a) shows the joint surplus between the entrepreneur and the insurer for a given contract $\hat{p}$. Panel (b) shows the trade-off to find the optimal contract: The gain from an improved bargaining position (Area A) versus the loss of bargaining surplus (Area B).

Therefore, when the insurer offers a contract that induces litigation, there is no loss in bargaining surplus (the negative term in Equation (8) disappears). Thus, the profit maximizing contract in that case is the one that maximizes the expected willingness to pay of consumers that will settle after they observe their type. Taking first order condition in Equation (8) for the case $\theta = 0$ we obtain:

$$\underbrace{(c + c_A) f(p^*)}_{\text{efficient coverage}} - \underbrace{\frac{c}{(p^*)^2} \int_{c/d}^{p^*} p dF(p)}_{\text{inefficient coverage}} .$$

By marginally increasing p^*, the monopolist changes the density of types that will receive efficient coverage—the type that will extract all the bargaining surplus in the negotiation with the third party. Types lower than p^* will settle but will not be able to capture all the bargaining surplus in the negotiation with the plaintiff—they will be receiving an inefficient amount of insurance, compared to the perfect information case.

In the case $\theta > 0$, there is a new force. The first order condition is given by

$$\underbrace{(1-\theta)(c+c_A)f(p^*)}_{\text{efficient settlement}} \underbrace{-(1-\theta)\frac{c}{(p^*)^2}\int_{c/d}^{p^*} p dF(p)}_{\text{inefficient settlement}} + \underbrace{\theta(c+c_A)f(p^*)}_{\text{bargaining surplus}} \tag{9}$$

By marginally increasing p^* some agents now settle and they are able to capture a bargaining surplus $\theta(c + c_A)$ in the settlement negotiation with the plaintiff instead of losing it by going to litigation (bargaining surplus term in Equation 9). Also, since the agent was already capturing part of the bargaining surplus in the negotiation, the benefit of an improved bargaining position is to recover the surplus captured by the plaintiff, which is a fraction $(1 - \theta)$ of the total bargaining surplus. The intuition for the first two terms is the same as the one given for the case $\theta = 0$.

These three forces are what the monopolist (or a perfectly competitive market) must balance to determine the contract that will be offered. If the solution is such that $F(p^*) = 1$, the contract that is offered does not induce ex-post litigation. This is going to depend on the shape of the distribution, the bargaining power of the agent, the litigation costs , and the size of the damages.

To gain more intuition, let us consider a distribution with only the two types.

Definition 1 (Two-types case)**.** *Let p_L and p_H such that $\frac{c}{d} < p_L < p_H \leq 1$. The probability distribution is $Pr(p = p_L) = \lambda_L$ and $Pr(p = p_H) = \lambda_H$.*

From Proposition 1 it is easy to see that for the two-types case the optimal contract can be only be either $p^* = p_L$ or $p^* = p_H$. Which one of these contracts is optimal depends on the fraction of types. When the proportion of high-risk types is relatively large (i.e., λ_H is large relative to λ_L), the optimal contract is $p^* = p_H$ and *targets* types p_H. Analogously, when λ_H is small the optimal contract is $p^* = p_L$.[13]

The idea that the shape of the distribution determines whether or not the optimal contract induces litigation can be exemplified by considering a particular family of distributions.[14]

Example 1. *The optimal monopoly contract for an uninformed agent is*

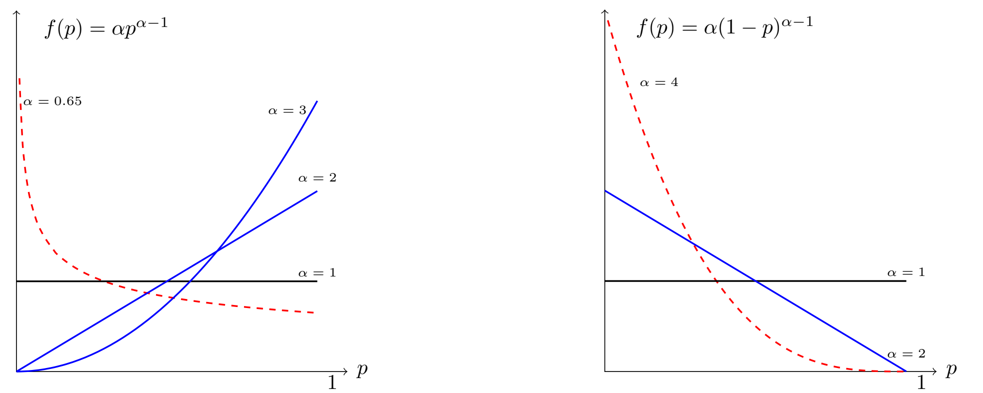
[13]The details of this case is in Appendix B.1.

[14]The details of the calculations to derive Example 1 are in Appendix A.1.

1. $p^* = 1$ *for the family of distributions* $F(p) = p^\alpha$, $\alpha > 0$.

2. $p^* < 1$ *for the family of distributions* $F(p) = 1 - (1-p)^\alpha$, $\alpha > 1$.

These families of distributions are illustrated in Figure 6. In Figure 6(a), the density allocates a significant portion of the mass to high-risk types, which is the reason why it is optimal to choose $p^* = 1$. In Figure 6(b), there is not too much mass in high-risk types, so it is optimal to set $p^* < 1$ and allow for some litigation.



(a) Family $F(p) = p^\alpha$ **(b)** Family $F(p) = 1 - (1-p)^\alpha$

Figure 6: (a) Density for the family $F(x) = x^\alpha$ for different values of α. (b) Density for the family $F(x) = 1 - (1-x)^\alpha$ for different values of $\alpha \geq 1$.

3.3 Adverse Selection

3.3.1 Perfectly Competitive Market: Equilibrium Analysis

Consider a market for insurance that is perfectly competitive and the agents are privately informed about the probability of liability at the time of signing the contract.

In modeling perfect competition, we follow Rothschild and Stiglitz (1976). There is a perfectly elastic supply of potential insurers capable of freely entering and selling insurance. Hence, insurer profit must be zero in equilibrium.

Given these conditions, the equilibrium price depends upon whether insurance induces litigation or settlement. If an insurance policy induces settlement for all the types that buy that policy, the price it must be zero in equilibrium, since then the insurer providing that insurance policy bears no cost. In contrast, if insurance induces litigation for some types, then Corollary 1 shows that the insurer earns a negative profit on the group of agents who litigate. Hence, to break even, the insurer must earn a strictly positive profit on the other group of agents. We will show that in a perfectly competitive market for third party insurance, a separating equilibrium does not exist, in contrast to a perfectly competitive market for first party insurance.

Proposition 2. *A single contract (pooling) that induces litigation cannot be offered in equilibrium in a perfectly competitive market.*

Proof. Consider a distribution of types $F \sim [0,1]$. If $F(p^*) < 1$, we will show that the contract p^* cannot be offered in equilibrium in a perfectly competitive market. Suppose p^* is offered in equilibrium at price P. Since $F(p^*) < 1$, then there is a positive mass of types that litigate, for which the insurer incur losses (Corollary 1). Then, in equilibrium, to break even insurers must be selling this contract at a positive price $P > 0$.

Consider an alternative contract $\tilde{p} = p^* + \varepsilon$ sold at price $\tilde{P}$, with ε sufficiently small. This new contract offers a lower damages coverage but it is cheaper. Contract $\tilde{p}$ is preferred by types $p < \tilde{p}$ over p^* and not preferred for types $p > \tilde{p}$ as long as

$$W(p, p^*) - P < W(p, \tilde{p}) - \tilde{P}, \quad \text{for all } p < \tilde{p}$$

and

$$W(p, p^*) - P > W(p, \tilde{p}) - \tilde{P}, \quad \text{for all } p > \tilde{p}$$

By Corollary 2, it is enough to have

$$\tilde{P} = P + W(\tilde{p}, \tilde{p}) - W(\tilde{p}, p^*) = P + c\left(\frac{p^* - \tilde{p}}{p^*}\right) = P - \frac{c}{p^*}\varepsilon.$$

Thus, for ε small enough, contract $\tilde{p}$ sold at price $\tilde{P} = P - \frac{c}{p^*}\varepsilon > 0$ attracts only types that settle and it is sold at a positive price. This profitable deviation implies that p^* cannot be offered in equilibrium. □

Proposition 2 shows that a single contract (pooling) that induces litigation cannot be offered in equilibrium in a perfectly competitive market. Intuitively, if such a contract were offered in equilibrium, an alternative contract sold at a positive price could be offered to attract types that settle, which does not impose any cost on the insurer. This intuition is similar to the *cream skimming* argument in Rothschild and Stiglitz (1976).

Theorem 1. *A separating equilibrium does not exist in a perfectly competitive market.*

Proof. We show it by contradiction. Let $\mathcal{M}$ be the set of contracts offered in equilibrium. In a separating equilibrium, at least two of these contracts must attract a different set of types. Let p_1^* and p_2^* with $p_1^* < p_2^*$, sold at prices P_1 and P_2, respectively, be such a pair of contracts. Let $D_i \subseteq [0,1]$ the set of types that prefer contract p_i^*, which is given by

$$D_i = \left\{ p \in \left[\frac{c}{d}, 1\right] \ : \ W(p, p_i^*) - P_i \geq W(p, p_j^*) - P_j, \quad \text{for all } p_j^* \in \mathcal{M} \right\}.$$

Let $D_i(S) = D_i \cap [0, p_i^*]$ and $D_i(L) = D_i \cap (p_i^*, 1]$ be the set of types that buy contract p_i^* and that settle and litigate, respectively. If the measure of the set $D_i(L)$ is zero, then $P_i = 0$, since the insurer would not bear any costs by offering p_i^*. But it cannot be that $D_1(L)$ and $D_2(L)$ have both measure zero, since they would be sold at price zero and by Corollary 2, types would pool at p_1^* (see Figure 4). This rules out separating equilibrium with any pair of contracts such that litigation is precluded under both, because such a pair would need to be priced at zero in equilibrium and types would pool at the lowest p_i^*. So, in any separating equilibrium we must have a positive measure of $D_i(L) > 0$ for some $i \in \{1, 2\}$. Without loss of generality, suppose that $\mu_F(D_1(L)) > 0$. Notice that if $\mu_F(D_1(S)) = 0$, then by Corollary 1 insurers incur losses by selling this contract. Thus, contract p_1^* must attract types that settle an must sell at a positive price $P_1 > 0$. Consider a new contract $\tilde{p}_1 = p^* + \varepsilon$ sold at price $\tilde{P}$ as in Proposition 2 to build a profitable deviation from p_1^*—by construction, this deviation only attracts types that settle. This profitable deviation implies that p_1^* cannot be offered in equilibrium, because then p_1^* would only attract types that litigate (it would be a money loser). This is a contradiction. □

The intuition of these result is easy to see with two types. Suppose agents can be low-risk (type p_1) or high-risk (type p_2), with $p_1 < p_2$. Any contract accepted by the low-risk type will also be accepted by the high-risk type, since for a fixed contract the willingness to pay is increasing in the liability probability. To separate types in equilibrium, there must be contracts offered at different prices. This implies that at least for one of those contracts some types must litigate, otherwise if every type settles the contracts must be offered at price zero. But if a contract only attracts types that litigate, Corollary 1 shows that the insurer incurs losses. Thus, the contract must attract types that settle and types that litigate. Since there are only two types, such a contract would pool all types and not separate them.

The result in Theorem 1 contrasts with Rothschild and Stiglitz (1976), where a separating equilibrium does exist provided there are a sufficiently high number of high-risk types. Also in contrast to Rothschild and Stiglitz (1976), we now show that a simple pooling equilibrium may exist in this market. From Proposition 2 and Theorem 1, the only possible equilibrium is a pooling equilibrium that does not induce litigation.

Theorem 2. *Let p^* such that $F(p^*) = 1$. A pooling equilibrium exists if and only if*

$$\max_{\tilde{p}\in\left[\frac{c}{d},p^*\right)} \left\{ \frac{(1-\theta)c\cdot(p^*-\tilde{p})}{\tilde{p}p^*} \cdot \max_{\bar{p}\in\left[\frac{c}{d},\tilde{p}\right]} \bar{p}[1-F(\bar{p})] - \int_{\tilde{p}^+}^{p^*} \left[c_A + \frac{cp}{\tilde{p}}\right] dF(p) \right\} \leq 0.$$

The pooling equilibrium contract is p^* sold at price zero.

Proof. By Proposition 2, there is no pooling equilibrium at p^* such that $F(p^*) < 1$. Hence, the only candidate is p^* such that $F(p^*) = 1$.

A contract $\tilde{p}$ sold a price $\tilde{P}$ is a profitable deviation if attracts enough low-risk types that settle but pay a positive price to compensate the loss of selling insurance to high-risk types that litigate and generate losses for the insurer. Let $\bar{p}$ the (unique by single crossing) type that is indifferent between $\tilde{p}$ at price $\tilde{P}$ and p^* for free. Then,

$$W(\bar{p}, p^*) = W(\bar{p}, \tilde{p}) - \tilde{P} \Rightarrow \tilde{P} = \bar{p}\left[\frac{(1-\theta)c\cdot(p^*-\tilde{p})}{\tilde{p}p^*}\right]$$

Next, we only consider contracts such that $\tilde{p} > \bar{p}$. In any other case, the insurer loses money by offering the deviation. Then, the profit of contract $\tilde{p}$ at price $\tilde{P}$ is given by

$$\tilde{P}[1-F(\bar{p})] - \int_{\tilde{p}}^{1} K(p,\tilde{p})dF(p) = \tilde{P}[1-F(\bar{p})] - \int_{\tilde{p}}^{1} \left[c_A + \frac{cp}{\tilde{p}}\right] dF(p)$$

Thus, we can think of choosing the best cutoff point $\bar{p}$ for a given $\tilde{p}$ and then choosing the best deviation $\tilde{p}$. Hence, there is no profitable deviation as long as the condition in the Theorem holds. □

Intuitively, the condition in Theorem 2 says that a pooling equilibrium exists as long as there is sufficient high-risk types so any deviation would induce such losses that it is not profitable to offer a contract that induces litigation. Consider the two-type case: There is a mass λ_H of high-risk types p^H and a mass $(1-\lambda)$ of low-risk types p^L. The candidate for pooling equilibrium is to sell contract $p^* = p^H$ to all types at price zero. This contract does not induce litigation. Applying the condition in Theorem 2, it is easy to see that the only deviation to consider is $\tilde{p} = \bar{p} = p^L$. Therefore, in this case the condition is equivalent to

$$\lambda_H \geq \frac{(1-\theta)c(p^H - p^L)p^L}{p^H(c_A p^L + cp^H)}.$$

When the population consists primarily of p^H types, then a free contract that targets these types is an equilibrium. The p^L types will also "buy" this contract. There is no way to "cream skim," because any better contract offered to p^L types also attracts too many litigious p^H types.

Theorems 2 and 1 in combination say that in a perfectly competitive market for liability insurance, only a pooling equilibrium can exist, and its existence will depend on the distribution of types. Notice the qualitative difference with the results in Rothschild and Stiglitz (1976). In their model, only a separating equilibrium exists under some conditions on the distribution of types, while a pooling equilibrium never exists.

3.3.2 Optimal Monopoly Contract With Adverse Selection

When agents have private information about their type, a monopolist insurer may offer a menu of contracts to maximize profits. By the revelation principle, we can restrict attention to direct mechanisms that are incentive compatible.

Our mechanism design problem, however, presents a subtle complication. For a given contract p^*, the willingness to pay and the cost for the monopolist are not differentiable at the point $p = p^*$. Carbajal and Ely (2013) study quasi-linear settings with non-differentiable valuations. In this case, the envelope theorem characterization may lead to a range of possible payoffs as a function of the allocation rule. The problem pointed out in Carbajal and Ely (2013) is that, although the valuation may be non-differentiable at one point (which has zero-measure), the mechanism may allocate a non-zero measure set of types to the non-differentiable point. The marginal valuation is not 'point-identified' at the non-differentiable point, because it belongs to an interval (the sub-differential instead of the derivative). However, in our context, we are able to show that the optimal mechanism allocates at most one type to the non-differentiable point, hence, the envelop theorem can be applied to derive the optimal mechanism. Before we present the main result of this section, we derive a series of results that are useful to characterize the optimal menu of contracts.

Instead of indexing contracts by $p^* \in \left[\frac{c}{d}, \infty\right]$, we define $x(p^*) = \frac{1}{p^*} \in \left[0, \frac{d}{c}\right]$ to be the allocation, which corresponds to $x \cdot c = \hat{\alpha}_D$, the amount of damages covered over the cost of the plaintiff. The insurer offers a direct revelation mechanism such that for each reported type p, the agent receives allocation $x(p)$ at price $T(p)$. Then, the payoff for an agent of type p that reports $\tilde{p}$ is given by:

$$U(p, \tilde{p}) = \hat{W}(p, x(\tilde{p})) - T(\tilde{p}),$$

where

$$\hat{W}(p, x) = \begin{cases} (1-\theta)(cpx + c_A) & px \leq 1, \\ cpx + c_A - \theta(c + c_A) & px > 1. \end{cases}$$

Notice that when $\theta = 0$, this is the classic quasilinear environment. When $\theta > 0$, the agent's payoff has a non-differentiable point (a kink) whenever $xp = 1$.

The insurer's cost of serving type p with allocation x is

$$K(p, x) = \begin{cases} 0 & px \leq 1, \\ cpx + c_A & px > 1. \end{cases}$$

The insurer's cost has a kink whenever $xp = 1$, regardless of the value of θ.

The problem of the insurer is to choose the functions $x(\cdot)$ and $T(\cdot)$ to solve:

$$\max_{P(\cdot),\ x(\cdot)} \int_{c/d}^{1} T(p) dF(p) - \int_{\{p: px(p) > 1\}} [c_A + cpx(p)] dF(p)$$

subject to

$$p \in \arg\max_{p'} \ \hat{W}(p, x(p')) - T(p') \tag{IC}$$

$$U(p,p) \geq 0 \tag{IR}$$

As is standard in the mechanism design literature, when the valuation satisfies supermodularity, the allocation features a monotonicity property.

Lemma 4. *In an incentive compatible mechanism $x(\cdot)$ must be non-decreasing.*

Lemma 4 shows that the supermodularity property of the willingness to pay implies that higher types must get more damages cover in order for the menu to be incentive compatible. The next lemma shows that given the non-decreasing property of the optimal allocation, there exists at most one type that receives the efficient amount of damages coverage.[15]

Lemma 5. *In the optimal menu of contract $px(p) = 1$ for at most one $p \in \left[\frac{c}{d}, 1\right]$.*

Proof. Suppose there exist $p_1 > p_2 > 0$ such that $p_1 x(p_1) = p_2 x(p_2) = 1$. Then, $x(p_1) = \frac{1}{p_1} < \frac{1}{p_2} = x(p_2)$. This is a contradiction, since the optimal menu of contracts is non-decreasing. □

Lemma 5 shows that, as an implication of incentive compatibility, at most one type will get efficient damages coverage. This result also allows us to use the envelope theorem and derive a unique payoff function for the optimal allocation, since the set of types for which the derivative of the payoff is not defined has measure zero for all incentive compatible contracts. The next lemma shows that the non-decreasing property of the optimal allocation implies that there must be threshold type, $\bar{p}$, that it is indifferent between settlement and litigation. Types below $\bar{p}$ will settle and types above $\bar{p}$ will litigate.

Lemma 6. *Suppose that in the optimal allocation $px(p) > 1$. Then, for $p' > p$ we must have $p'x(p') > 1$.*

Proof. Suppose that $p' > p$, $px(p) > 1$, and that (by contradiction) $p'x(p') \leq 1$. Then, $p'x(p') < px(p)$. But this is a contradiction since $x(p') \geq x(p)$ in the optimal contract. □

[15]Interestingly, this will not be in general the type 'at the top', but the type at the 'kink.'

Lemmas 5 and 6 allows us to characterize the optimal contract as a threshold strategy: There exits $\hat{p} \in \left[\frac{c}{d}, 1\right]$ such that for all types $p \leq \hat{p}$ there is settlement and for types $p > \hat{p}$ there is litigation. With these results, we can now characterize the optimal monopoly contract, restricting attention to *regular* distributions, i.e., when $p - \frac{1-F(p)}{f(p)}$ is increasing in p.

Theorem 3. *Let $\bar{p}$ be the solution to $\bar{p} = \frac{1-F(\bar{p})}{f(\bar{p})}$. Define p^* as*

$$p^* \in \underset{\hat{p}\in[\hat{p},\infty]}{\arg\max}\ (1-\theta)c_A F(\bar{p}) + (1-\theta)\int_{\bar{p}}^{\hat{p}} \left[c_A + \frac{c}{\hat{p}}\left(p - \frac{1-F(p)}{f(p)}\right)\right] f(p)dp$$

$$- \int_{\hat{p}}^{1} \left[\theta(c+c_A) + \frac{c}{\hat{p}}\left(\frac{1-F(p)}{f(p)}\right)\right] f(p)dp.$$

The optimal menu of contracts offered by a monopolist insurer consist of (at most) two contracts:

1) $(c_A, 0)$ sold at price $T(p) = (1-\theta)c_A$ for types $p \leq \bar{p}$;

2) Contract $\left(c_A, \frac{c}{p^*}\right)$ sold at price $T(p) = (1-\theta)\left(c_A + c\frac{\bar{p}}{p^*}\right)$ for types $p > \bar{p}$.

Proof. See Appendix ☐

Theorem 3 characterizes the optimal contract for a regular distribution. First, we find a type $\bar{p}$ that separates types with positive and negative virtual surplus. Unlike the standard setting, where the mechanism excludes types with negative surplus, in our setting 'exclusion' refers to not covering damages. The insurer can always offer the contract that only covers litigation costs, whose value is independent of types and therefore does not introduce information rents for higher types. The price of that contract is $(1-\theta)c_A$, i.e., the insurer extract all the surplus of agents buying this contract and receives a profit of $(1-\theta)c_A F(\bar{p})$, which is the first term in the expression in the theorem.

For types above $\bar{p}$ the insurer wants to offer a contract that covers damages which corresponds to the efficient contract for some type $\hat{p}$. Relative to the efficient provision of insurance, described in Section 3.2, the monopolist's contract distorts the incentives in two different ways. First, for types in $[\bar{p}, \hat{p}]$, they settle but receive less insurance compared to the first best. Second, for types in $[\hat{p}, 1]$ the contract induces agents to litigate, which generates a loss of $\theta(c+c_A)$ in surplus. By lowering $\hat{p}$ the insurer increases the willingness to pay of all agents, but induces litigation for a larger set of types. The optimal damages contract, denoted by p^*, captures this trade-off.

Notice the similarity in expected profit of the monopolist under adverse selection and the case of an uninformed agent. To satisfy incentive compatibility, the insurer must give informational rents to the agents, captured by the term $\frac{1-F(p)}{f(p)}$. The trade off is similar to that of Proposition 1, except now the insurer must consider information rents and the fact that not everyone is getting the same policy.

3.4 Litigation Frequency with and without Adverse Selection

We study how much litigation is induced by either a perfectly competitive market or a monopolist under adverse selection. Notice that with complete information the first best contract never induces litigation. In contrast, with incomplete information the contracts offered in equilibrium may induce litigation.

First, in a perfectly competitive market it is easy to see that adverse selection induces less litigation than a setting in which there is symmetric information between the agent and the insurer. From Proposition 1, the contract offered by a perfectly competitive market may induce litigation, as shown in Example 1. However, when an equilibrium with adverse selection exists, as shown in Proposition 2 the only possibility is a pooling equilibrium at the top of the distribution, i.e., litigation does never occurs.

Second, consider a monopolist insurer. To compare the level of litigation with symmetric and asymmetric information, we need to compare the solution to the problem in Proposition 1 and Theorem 3.

Proposition 3. *The optimal monopoly contract with symmetric information induces weakly more litigation then the contract under asymmetric information.*

These results show the amount of litigation increases with the lack of information held by the insurer. Figure 7 summarizes this finding.

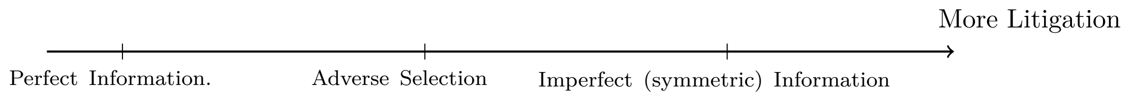


Figure 7: Amount of litigation induced in equilibrium depending how informed are the insurers.

4 Conclusion

We study third-party (liability) insurance contracts under adverse selection and ex-post moral hazard. As a result, equilibrium contracts in third-party insurance markets are quite different than those in the market for first-party insurance. In a perfectly competitive market for third-party insurance, only pooling equilibrium can exist, in contrast to Rothschild and Stiglitz (1976) where only separating equilibrium can exist. Separating equilibrium does not exist in our setting because any contract induces some types to settle, imposing no cost to the insurer, and others types to litigate, which is costly for the insurer (and inefficient). In a separating equilibrium at least one contract attract both types that settle and types litigate. However, types that settle can be "cream skimmed" by offering an alternative contract.

With a monopolist insurer, the optimal contract is qualitatively different from first party insurance studied by Stiglitz (1977) and Chade and Schlee (2012). First, we find that the optimal contract may distort types "at the top"—for some distributions, only an interior type gets efficient insurance. Second, while the problem can be solved using techniques from mechanism design, because the agents are risk-neutral, our result differs from the classic discriminating monopolist under private information (Mussa and Rosen, 1978). Given the particular characteristics of the insurer's cost function and the willingness to pay of the agent, there are points of non-differentiability affecting the shape of the optimal contract (Carbajal and Ely, 2013).

Our paper is a step forward to better understanding contracts in third-party insurance markets. Our findings motivate further study of additional characteristics of liability insurance. First, we assume settlement negotiations follow complete-information Nash bargaining, where settlement occurs if and only if there is a non-negative bargaining surplus. This is a key factor in the existence of pooling equilibria under competition, because insurance contracts in that equilibrium carry the same (zero) costs for all types. Under multiple alternative settlement bargaining models, such as screening (Bebchuk, 1984) or signaling (Reinganum and Wilde, 1986), litigation may occur in equilibrium, and with a different probability for different types of agents. Second, motivated by patent litigation insurance, we assume that the agent controls the lawsuit. For certain other types of liability insurance, it is customary to assign control to the insurance company (Meurer, 1992). It would interesting to consider control as a choice for contracting, as in Aghion and Bolton (1992) and Bolton and Oehmke (2011), and to understand how control affects the value of liability insurance. Third, we restrict attention to risk-neutral agents. This assumption has the virtue of highlighting the different value proposition of liability insurance. However, some purchasers of liability insurance (especially small businesses) may pursue liability insurance in part to mitigate risk. If equilibrium contracts rule out litigation, then the presence of risk aversion does not alter the result that the insurer's costs are the same across types. However, it could alter the willingness to pay for insurance, which might have implications for equilibrium menus of contracts under monopoly.

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A Appendix A: Proofs

Proof of Lemma 1

Proof. Given that there is always settlement without insurance, the willingness to pay for insurance is

$$W(p,\alpha) = \begin{cases} V(p,\alpha,S) - V(p,0,S) & \text{if } p \leq p^* \\ V(p,\alpha,L) - V(p,0,S) & \text{if } p > p^* \end{cases}$$

where

$$V(p,\alpha,S) = -c_A - pd + \theta(c + c_A) + (1-\theta)(\widehat{\alpha}_L + p\widehat{\alpha}_L),$$

$$V(p,\alpha,L) = -c_A - pd + \widehat{\alpha}_L + p\widehat{\alpha}_D.$$

Then, we can write

$$W(p,\alpha) = \widehat{\alpha}_L + p\widehat{\alpha}_D - \theta \cdot \begin{cases} \widehat{\alpha}_L + p\widehat{\alpha}_D & \text{if } p \leq p^* \\ (c + c_A) & \text{if } p > p^* \end{cases}$$

Notice that $\widehat{\alpha}_D p^* = c + c_A - \widehat{\alpha}_L$, so adding and subtracting $p^*\widehat{\alpha}_D$ we can write:

$$\widehat{\alpha}_L + p\widehat{\alpha}_D = \widehat{\alpha}_L + p^*\widehat{\alpha}_D - p^*\widehat{\alpha}_D + p\widehat{\alpha}_D = c + c_A + (p - p^*)\widehat{\alpha}_D$$

Thus, we can write:

$$W(p,\alpha) = (1-\theta)(c + c_A) + (p - p^*)\widehat{\alpha}_D \begin{cases} (1-\theta) & \text{if } p \leq p^* \\ 1 & \text{if } p > p^* \end{cases}$$

The insurer's cost is given by

$$K(p,\alpha) = \begin{cases} 0 & \text{if } p \leq p^* \\ \widehat{\alpha}_L + p\widehat{\alpha}_D & \text{if } p > p^* \end{cases}$$

or

$$K(p,\alpha) = \begin{cases} 0 & \text{if } p \leq p^* \\ c + c_A + (p - p^*)\widehat{\alpha}_D & \text{if } p > p^* \end{cases}$$

□

Proof of Corollary 2

Proof. Consider $p' > p$. Let $g(p^*) = W(p', p^*) - W(p, p^*)$. Then, we have:

$$g(1-p^*) = c\left(\frac{p'-p}{p^*}\right) - \begin{cases} 0 & p^* < p \\ c\theta\left(\frac{p^*-p}{p^*}\right) & p \le p^* < p' \\ c\theta\left(\frac{p'-p}{p^*}\right) & p^* \ge p' \end{cases}$$

It is easy to see that $g(p^*)$ is decreasing in p^*. Therefore, $\tilde{g}(p^*) = g(1-p^*)$ is increasing in p^* which implies that $\tilde{W}$ is supermodular. □

Proof of Proposition 1

Proof. Consider Equation (7) and replace the expressions in equations 5 and 6. Notice that

$$W(p, p^*) - K(p, p^*) \quad = \quad \begin{cases} (1-\theta)\left[c_A + c\dfrac{p}{p^*}\right] & \text{if } p \le p^* \\ -\theta(c + c_A) & \text{if } p > p^* \end{cases}.$$

Taking expected value over p we get the expression in the proposition. □

Proof of Lemma 4

Proof. Consider types $p_1 > p_2$. Then, incentive compatibility implies

$$W(p_1, x(p_1)) - T(p_1) \ge W(p_1, x(p_2)) - T(p_2)$$

and

$$W(p_2, x(p_2)) - T(p_2) \ge W(p_2, x(p_1)) - T(p_1)$$

Adding up these two inequalities we get:

$$W(p_1, x(p_1)) - W(p_2, x(p_1)) \ge W(p_1, x(p_2)) - W(p_2, x(p_2)).$$

Let $g(x) = W(p_1, x) - W(p_2, x)$. It is easy to see (Corollary 2) that

$$g(x) = \begin{cases} c(1-\theta)(p_1 - p_2)x & xp' \leq 1 \\ c(p_1 - p_2)x + \theta c(px - 1) & px \leq 1 < p'x \\ c(p_1 - p_2)x & xp' \leq 1 \end{cases}$$

is an strictly increasing function. Therefore, incentive compatibility is equivalent to

$$g(x(p_1)) \geq g(x(p_2)) \Rightarrow x(p_1) \geq x(p_2).$$

□

Proof of Theorem 3

Proof. Let p be the agent's type and $x(p) \in [0, d/c]$ be the allocation. Consider a direct revelation mechanism: $p \to (x(p), T(p))$, where $x(\cdot)$ and $T(\cdot)$ are the allocation and price for an agent who reports type p. The problem of the insurer is to choose $x(\cdot)$ and $T(\cdot)$ to solve:

$$\max_{T(\cdot), x(\cdot)} \int_{c/d}^{1} T(p) dF(p) - \int_{\{p: px(p) > 1\}} [c_A + cpx(p)] dF(p)$$

subject to

$$p \in \arg\max_{p'} \underbrace{\hat{W}(p, x(p')) - T(p')}_{\equiv u(p,p')} \qquad \text{(IC)}$$

Let $V(p) = \max_{p'} u(p, p')$. By the envelope theorem and incentive compatibility we have:

$$V'(p) = \begin{cases} (1-\theta)cx(p) & px(p) \leq 1 \\ cx(p) & px(p) > 1 \end{cases}$$

By Lemma 4, $x(\cdot)$ must be weakly increasing for incentive compatibility. Hence $px(p)$ is strictly increasing, and therefore, there exists some (unique) type $\hat{p}$ such that $px(p) > 1$ for all $p > \hat{p}$ and $px(p) \leq 1$ for all $p \leq \hat{p}$ (though it may be that $\hat{p} = 1$).

Now we have, for $p \leq \hat{p}$:

$$V(p) = V(c/d) + \int_{c/d}^{p} (1-\theta)cx(s) ds$$

and for $p > \hat{p}$:

$$V(p) = V(c/d) + \int_{c/d}^{\hat{p}} (1-\theta)cx(s)ds + \int_{\hat{p}}^{p} cx(s)ds$$

Since $V(p) = u(p,p)$ we have that for $p \leq \hat{p}$,

$$T(p) = (1-\theta)(cpx(p) + c_A) - V(c/d) - \int_{c/d}^{p} (1-\theta)cx(s)ds$$

and for $p > \hat{p}$,

$$T(p) = cpx(p) + c_A - \theta(c + c_A) - V(c/d) - \int_{c/d}^{\hat{p}} (1-\theta)cx(s)ds - \int_{\hat{p}}^{p} cx(s)ds$$

Note that it is optimal to set $V(c/d) = 0$. The monopolist's problem, for any $\hat{p}$, is:

$$\begin{aligned}
&\max_{x(\cdot)} \int_{c/d}^{\hat{p}} \left[(1-\theta)(cpx(p) + c_A) - \int_{c/d}^{p} (1-\theta)cx(s)ds\right] dF(p) + \\
&\quad + \int_{\hat{p}}^{1} \left[cpx(p) + c_A - \theta(c + c_A) - \int_{c/d}^{\hat{p}} (1-\theta)cx(s)ds - \int_{\hat{p}}^{p} cx(s)ds\right] dF(p) - \\
&\quad - \int_{\hat{p}}^{1} [cpx(p) + c_A]\, dF(p) \\
\Leftrightarrow\quad &\max_{x(\cdot)} \int_{c/d}^{\hat{p}} \left[(1-\theta)(cpx(p) + c_A) - \int_{c/d}^{p} (1-\theta)cx(s)ds\right] dF(p) + \\
&\quad + \int_{\hat{p}}^{1} \left[-\theta(c + c_A) - \int_{c/d}^{\hat{p}} (1-\theta)cx(s)ds - \int_{\hat{p}}^{p} cx(s)ds\right] dF(p)
\end{aligned}$$

To simplify this expression, with some algebra we can show the following:

$$\int_{c/d}^{\hat{p}} \int_{c/d}^{p} (1-\theta)cx(s)f(p)dsdp = \int_{c/d}^{\hat{p}} \int_{s}^{\hat{p}} (1-\theta)cx(s)f(p)dpds = \int_{c/d}^{\hat{p}} (1-\theta)cx(p)\frac{F(\hat{p}) - F(p)}{f(p)} dF(p)$$
$$\int_{\hat{p}}^{1} \int_{c/d}^{\hat{p}} (1-\theta)cx(s)f(p)dsdp = \int_{c/d}^{\hat{p}} \int_{\hat{p}}^{1} (1-\theta)cx(s)f(p)dpds = \int_{c/d}^{\hat{p}} (1-\theta)cx(p)\frac{1 - F(\hat{p})}{f(p)} dF(p)$$
$$\int_{\hat{p}}^{1} \int_{\hat{p}}^{1} cx(s)f(p)dsdp = \int_{\hat{p}}^{1} \int_{s}^{1} cx(s)f(p)dpds = \int_{\hat{p}}^{1} cx(p)\frac{1 - F(p)}{f(p)} dF(p)$$

Now we can re-write the problem as:

$$\begin{aligned}
&\max_{x(\cdot)} \int_{c/d}^{\hat{p}} \left[(1-\theta)cx(p)\left(p - \frac{1 - F(p)}{f(p)}\right)\right] dF(p) - \int_{\hat{p}}^{1} \left[cx(p)\left(\frac{1 - F(p)}{f(p)}\right)\right] dF(p) + \\
&\quad + \int_{c/d}^{\hat{p}} c_A dF(p) - \int_{\hat{p}}^{1} \theta c dF(p) - \theta c_A
\end{aligned}$$

where the final three terms do not depend on $x(\cdot)$.

Let $\bar{p}$ such that $\bar{p} = \frac{1-F(\bar{p})}{f(\bar{p})}$. Notice that the optimal mechanism, by definition, must have $x(\hat{p}) = \frac{1}{\hat{p}}$. For $p > \hat{p}$, the objective function is decreasing in $x(p)$, and given that $x(p)$ is weakly increasing to satisfy incentive compatibility, it is optimal to set $x(p) = x(\hat{p})$. For $p \leq \hat{p}$ there are two cases: 1) If $p \leq \bar{p}$, we set $x(p) = 0$. Notice that this does not restrict the monotonocity condition for higher values of p; s) If $\bar{p} \leq \hat{p}$, then for $\bar{p} < p \leq \hat{p}$, we would like to make $x(p)$ as large as possible. However, since incentive compatibility imposes that $x(p)$ must be weakly increasing and $x(\hat{p}) = \frac{1}{\hat{p}}$, the best the insurer can do is to set $x(p) = x(\hat{p})$. Finally, if $\hat{p} < \bar{p}$ we would set $x(p) = 0$ for all p. It is easy to see that setting $\hat{p} < \bar{p}$ is not optimal. Then, to satisfy incentive compatibility, the optimal contract we must have:

$$x(p) = \begin{cases} 0 & \text{if } p \leq \bar{p}, \\ \dfrac{1}{\hat{p}} & \text{otherwise.} \end{cases}$$

Hence, to determine the optimal contract, the insurer chooses $\hat{p}$ to solve:

$$\max_{\hat{p} \in [\bar{p}, \infty]} (1-\theta) c_A F(\bar{p}) + (1-\theta) \int_{\bar{p}}^{\hat{p}} \left[c_A + \frac{c}{\hat{p}} \left(p - \frac{1-F(p)}{f(p)} \right) \right] f(p) dp$$
$$- \int_{\hat{p}}^{1} \left[\theta(c + c_A) + \frac{c}{\hat{p}} \left(\frac{1-F(p)}{f(p)} \right) \right] f(p) dp.$$

□

Proof of Proposition 3

Proof. Denote by p^*_S the optimal contract in Proposition 1 and let p^*_{AS} the optimal contract in Theorem 3.[16] Denote by $\bar{p}$ the solution to $\bar{p} = \frac{1-F(\bar{p})}{f(\bar{p})}$,and let H_S the objective function in Proposition 1, i.e.,

$$H_S(\hat{p}) = (1-\theta) \int_{c/d}^{\hat{p}} \left[c_A + \frac{cp}{\hat{p}} \right] dF(p) - \theta(c + c_A)[1 - F(\hat{p})].$$

[16]For simplicity, we can assume that the solution of each of these problems is unique. If not, our conclusion holds under the notion of strong set order.

Notice that p_S^* belongs to the interval $[\frac{c}{d}, 1]$ and, with a regular distribution, $\bar{p} \leq p_{AS}^*$ $\bar{p}$. Thus, whenever $p_S^* \leq \bar{p}$ we have $p_S^* \leq p_{AS}^*$.

Consider the case $p_S^* \geq \bar{p}$. Then,

$$p_S^* \in \arg\max_{\hat{p} \in \left[\frac{c}{d}, \infty\right]} H_S(\hat{p}) = \arg\max_{\hat{p} \in [\bar{p}, \infty]} H_S(\hat{p}).$$

It is easy to see that the objective function in Theorem 3 can be written as $H_S(\hat{p}) - \Delta(\hat{p})$, where

$$\Delta(\hat{p}) = \frac{(1-\theta)c}{\hat{p}} \int_{c/d}^{\bar{p}} p f(p) dp + \frac{(1-\theta)c}{\hat{p}} \int_{\bar{p}}^{\hat{p}} (1 - F(p)) dp + \frac{c}{\hat{p}} \int_{\hat{p}}^{1} (1 - F(p)) dp.$$

Consider the problem

$$p^*(\beta) = \arg\max_{\hat{p} \in [\bar{p}, \infty]} H_S(\hat{p}) - \beta \Delta(\hat{p}),$$

so $p^*(0) = p_S^*$ and $p^*(1) = p_{AS}^*$. By Topkis theorem, when $\Delta'(\hat{p}) < 0$ for all $\hat{p}$ we have $p^*(0) \leq p^*(1)$. Notice that

$$\Delta(\hat{p}) = \frac{(1-\theta)c}{\hat{p}} \left[\int_{c/d}^{\bar{p}} p f(p) dp + \int_{\bar{p}}^{1} (1 - F(p)) \right] + \theta \frac{c}{\hat{p}} \int_{\hat{p}}^{1} (1 - F(p)) dp.$$

Denote by A the expression in the bracket, which is independent of $\hat{p}$. Then, taking derivative we get

$$\Delta'(\hat{p}) = -\frac{c}{\hat{p}^2} \left[(1-\theta)A + \theta \int_{\hat{p}}^{1} (1 - F(p)) dp \right] - \theta \frac{c}{\hat{p}} (1 - F(\hat{p})) < 0.$$

□

A.1 Details of Example 1

Derivation for $F(x) = x^\alpha$

Consider the family of distributions:

$$F(x) = x^\alpha, \quad \alpha > 0.$$

When $\alpha = 1$ this is the uniform distribution. The objective function in the maximization problem can be written as:

$$(1-\theta)c_E[F(\hat{p}) - F(L)] + (1-\theta)\frac{c}{\hat{p}} \int_{L}^{\hat{p}} p \alpha p^{\alpha-1} dp - \theta(c + c_E)(1 - \hat{p}^\alpha)$$

$$(1-\theta)c_E(\hat{p})^\alpha + (1-\theta)\frac{c}{\hat{p}}\alpha\frac{(\hat{p})^{\alpha+1}-L^{\alpha+1}}{\alpha+1} + \theta(c+c_E)\hat{p}^\alpha$$

$$(\hat{p})^\alpha\left[(1-\theta)c_E + (1-\theta)c\frac{\alpha}{\alpha+1} + \theta(c+c_E)\right] - (1-\theta)\frac{c}{\hat{p}}\alpha\frac{L^{\alpha+1}}{\alpha+1}$$

$$(\hat{p})^\alpha\left[c_E + c\left(\frac{\alpha+\theta}{\alpha+1}\right)\right] - \frac{\alpha c(1-\theta)L^{\alpha+1}}{\alpha+1}\frac{1}{\hat{p}}$$

Hence, to find the optimal contract we must solve

$$\max_x \ f(x) \equiv Ax^\alpha - \frac{B}{x},$$

where

$$A = \left[c_E + c\left(\frac{\alpha+\theta}{\alpha+1}\right)\right] > 0, \quad B = \frac{\alpha c(1-\theta)L^{\alpha+1}}{\alpha+1} \geq 0.$$

Notice that

$$f'(x) = A\alpha x^{\alpha-1} + \frac{B}{x^2},$$

so $f' > 0$ for all x. Hence, the solution is to choose $x^* = 1$, which is independent of the value of α.

Derivation for $F(x) = 1-(1-x)^\alpha$ Consider the family of distributions:

$$F(x) = 1-(1-x)^\alpha, \quad \alpha > 1.$$

When $\alpha = 1$ this is the uniform distribution. The density is given by $f(x) = \alpha(1-x)^{\alpha-1}$. The objective function in the maximization problem can be written as:

$$(1-\theta)c_E[F(\hat{p}) - F(L)] + (1-\theta)\frac{c}{\hat{p}}\int_L^{\hat{p}} pf(p)dp - \theta(c+c_E)(1-F(\hat{p}))$$

Ignoring constant terms, we can write

$$-(1-\hat{p})^\alpha(c_E+\theta c) + \frac{c(1-\theta)\alpha}{\hat{p}}\int_L^{\hat{p}} p(1-p)^\alpha dp$$

For the integral we do a change of variables

$$\int_L^{\hat{p}} p(1-p)^{\alpha-1}dp = \int_{1-L}^{1-\hat{p}}(1-u)u^{\alpha-1}(-du) = \frac{1}{\alpha+1}\left[(1-L)^\alpha(1+\alpha L) - (1-\hat{p})^\alpha(1+\alpha\hat{p})\right]$$

The problem is equivalent to

$$\max_x f(x) \equiv -(1-x)^\alpha[A + \frac{\tilde{B}}{x}] + \frac{D}{x}$$

where

$$A = \left[c_E + c\left(\frac{\alpha+\theta}{\alpha+1}\right)\right] > 0, \quad \tilde{B} = \frac{c(1-\theta)}{\alpha+1} \geq 0, \quad D = \frac{c(1-\theta)(1-L)^\alpha(1+\alpha L)}{\alpha+1} \geq 0.$$

Notice that

$$f'(x) = \alpha(1-x)^{\alpha-1}[A + \frac{\tilde{B}}{x}] + \frac{(1-x)^\alpha \tilde{B}}{x^2} - \frac{D}{x^2}$$

For $\alpha > 1$, evaluating at $x = 1$ we have that $f'(1) = -D \leq 0$. This means the objective function is decreasing at $x = 1$. In fact, for $x \in [1-\varepsilon, 1]$ we have $f'(x) < 0$. Therefore, the maximum for $x \in [x/d, 1]$ is strictly less than 1.

B Appendix B: Extensions

B.1 Optimal Contract with Symmetric Information for the Two-Type case

Policy $p^* = \infty$ only covers litigation costs. With this policy all the agents settle and the expected profit for a monopolist is $\pi_\infty = (1-\theta)c_A$. This corresponds to the fraction of bargaining surplus that agents are able to capture from their improved bargaining position.

Policy $p^* = p_H$ is efficient for type p_H but inefficient for type p_L. By Proposition 1, the profit of this policy is

$$\pi_H = (1-\theta)(c+c_A)\lambda_H + (1-\theta)\left[c_A + c\frac{p_L}{p_H}\right]\lambda_L,$$

which corresponds to the efficient surplus for type p_H (first term), plus the efficient surplus for type p_L minus a reduction from inefficient coverage (p_H) to types p_L, leading to inefficient settlement (term in the bracket).

Policy $p^* = p_L$ is efficient for type p_L. There are no lower types, so there is no inefficient settlement. However, types above p_L are engaging in litigation and therefore losing $\theta(c + c_A)$, the bargaining surplus they used to capture in the settlement negotiation.

By Proposition 1, the profit of this policy is then

$$\pi_L = (1-\theta)(c+c_A)\lambda_L - \theta(c+c_A)\lambda_H.$$

The first term is the efficient bargaining surplus captured by types p_L, and the second term is the bargaining surplus lost by types p_H because they litigate instead of settling.

We can see that $\pi_H > \pi_\infty$ so that contract is dominated. The optimal contract will now depend on the relative mass of high-risk types.

Corollary 3. *For the two-types case, there exists a threshold λ^* such that for $\lambda_H \geq \lambda\lambda$ the optimal contract is $p^* = p_H$ and for $\lambda_H < \bar{\lambda}$ the optimal contract is $p^* = p_L$*

Proof. Writing π_H and π_L as a function of λ_H we find that π_H increases and π_L is decreases. We can find $\bar{\lambda}$ such that $\pi_H(\bar{\lambda}) = \pi_L(\bar{\lambda})$ where

$$\bar{\lambda} = \frac{(1-\theta)c(p_H - p_L)}{p_H(c+c_A) + (1-\theta)c(p_H - p_L)}.$$

□

B.2 Covering Settlement

Consider a contract that not only covers the legal costs and the damages, but also covers the settlement payment up to an amount $\hat{\alpha}_S$. Thus, a contract is now defined by three parameters: $\alpha = (\hat{\alpha}_L, \hat{\alpha}_D, \hat{\alpha}_S)$. There are several possible outcomes: Going to court or agreeing on a settlement fee ϕ. Suppose the agent and the third party agree on a settlement fee ϕ. Then, the payoff of the agent and the third party are

$$u_A = \min\{\hat{\alpha}_S - \phi, 0\}, \quad u_{TP} = \phi,$$

respectively. The joint surplus for this agreement is $J = \min\{\hat{\alpha}_S, \phi\}$, which is weakly increasing in ϕ. Therefore, the best arrangement between the agent and the third party is to set $\phi = \hat{\alpha}_S$.[17] Notice that in this case, $J = \hat{\alpha}_S$.

[17] This is without loss of generality since setting $\phi > \alpha_S$ does not increase the joint surplus

The disagreement payoff is to go to court. In that case, the third party gets $pd - c$ and the agent gets $-pd - c_A + \hat{\alpha}_L + p\hat{\alpha}_D$.

The increase in joint surplus for an agreement is

$$S_B = \hat{\alpha}_S + c + c_A - \hat{\alpha}_L - p\hat{\alpha}_D$$

Thus, the joint surplus between the agent and the third party from settling is larger than the joint surplus from going to court if and only if

$$S_B = \hat{\alpha}_S + c + c_A - \hat{\alpha}_L - p\hat{\alpha}_D \geq 0 \tag{10}$$

Types p below the threshold p^* settle, where

$$p \leq p^* \equiv \frac{\hat{\alpha}_S + c + c_A - \hat{\alpha}_L}{\hat{\alpha}_D} \tag{11}$$

Then, the third party gets a payoff equal to

$$u_{TP}^{\text{agreement}} = pd - c + (1-\theta)S_B$$

This payoff must equal the payoff of the agreement outcome $\hat{\alpha}_S + T$, so we have:

$$T = pd - c - \hat{\alpha}_S + (1-\theta)[\hat{\alpha}_S + c + c_A - \hat{\alpha}_L - p\hat{\alpha}_D]$$

$$T = pd - c - \theta\hat{\alpha}_S + (1-\theta)[c + c_A - \hat{\alpha}_L - p\hat{\alpha}_D]$$

Notice that, compared to the case in which the insurance company does not pay for settlement, the agent pays a lower fee when settling. Hence, there are two effect: The threshold for settlement changes, and the agent pays a lower settlement fee when settling.

The value of insurance is then,

$$W(\alpha) = \begin{cases} \theta\hat{\alpha}_S + (1-\theta)(\hat{\alpha}_L + p\hat{\alpha}_D) & p \leq p^* \\ (\hat{\alpha}_L + p\hat{\alpha}_D) - \theta(c + c_A) & p > p^* \end{cases}$$

The cost for the insurer from offering a contract α is:

$$K(\alpha) = \begin{cases} \hat{\alpha}_S & p \leq p^* \\ (\hat{\alpha}_L + p\hat{\alpha}_D) & p > p^* \end{cases}$$

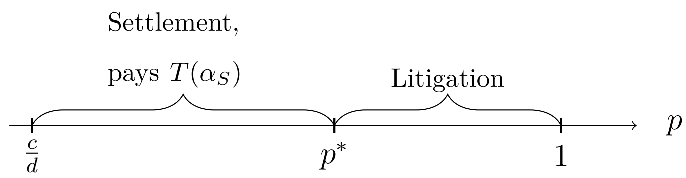

Figure 8: The effect of insurance contract α on licensing and litigation for different types of agents.

$$W(\alpha) = \begin{cases} \hat{\alpha}_S + (1-\theta)(c + c_A) + (1-\theta)\hat{\alpha}_D(p - p^*) & p \leq p^* \\ \hat{\alpha}_S + (1-\theta)(c + c_A) + \hat{\alpha}_D(p - p^*) & p > p^* \end{cases}$$

The cost for the insurer from offering a contract α is:

$$K(\alpha) = \begin{cases} \hat{\alpha}_S & p \leq p^* \\ \hat{\alpha}_S + (c + c_A) + \hat{\alpha}_D(p - p^*) & p > p^* \end{cases}$$

We can see that the joint surplus is independent of $\hat{\alpha}_S$. In fact, the solution is the same as in the baseline model setting $\hat{\alpha}_S = 0$.

Lemma 7. *Paying for settlement is never optimal, i.e., $\hat{\alpha}_S = 0$.*

Proof. It is easy to see that $\hat{\alpha}_L = c_A$ in the optimal contract. Consider $\hat{\alpha}_S > 0$ and $\hat{\alpha}_D$ that induce some threshold $p^* = \frac{\hat{\alpha}_S + c}{\hat{\alpha}_D}$. Consider a new contract, $\hat{\alpha}'_S = 0$ and $\hat{\alpha}'_D < \hat{\alpha}_D$ such that $p^* = \frac{c}{\hat{\alpha}'_D}$. Notice that $W(\alpha) - K(\alpha)$ is decreasing in $\hat{\alpha}_D$ for $p \leq p^*$ and independent of $\hat{\alpha}_D$ for $p > p^*$. Moreover, $W(\alpha) - K(\alpha)$ is independent of $\hat{\alpha}_S$. Therefore, the solution conditional on any particular p^* is the contract with the lower $\hat{\alpha}_D$. □

B.3 Settlement Under Signaling or Screening

B.3.1 Bebchuk

Suppose that the patentee makes a take-it-or-leave-it offer to the agent to settle the case. An agent of type p has the expected payoff from going to court, conditional on owning insurance α, of

$$\Pi - (c_A - \widehat{\alpha}_L) - p(d - \widehat{\alpha}_D).$$

If the patentee makes settlement offer S, then if the patentee accepts the offer, it earns $\Pi - S$. Hence, if the patentee knows the agent's type, the optimal settlement offer is

$$S^* = (c_A - \widehat{\alpha}_L) + p(d - \widehat{\alpha}_D).$$

Comparing this to the payoff without insurance, we find that the agent would be willing to pay

$$\widehat{\alpha}_L + p\widehat{\alpha}_D$$

for this contract.

Now consider the case of private information, where there are two types of agents. There are only three possible optimal settlement offers. First, the *weak* offer holds the low-risk type to reservation utility:

$$S^L = (c_A - \widehat{\alpha}_L) + p^L(d - \widehat{\alpha}_D).$$

If the patentee makes this offer, both types accept. The high-risk type, who would have been willing to pay more, receives some rent.

Second, the *aggressive* offer holds the high-risk type to reservation utility:

$$S^H = (c_A - \widehat{\alpha}_L) + p^H(d - \widehat{\alpha}_D).$$

If the patentee makes this offer, the low-risk types decline the offer and pursue litigation. The high-risk types accept the offer.

Third, the patentee may make a *bad faith* offer, of $S > S^H$. Under this offer, both types choose litigation. The patentee will make this offer only if it strictly prefers litigation. Intuitively, this could occur if the agent has so much insurance that there is no bargaining surplus.

Let's focus attention on the case where the patentee makes one of the two good-faith offers. For the patentee, if offer S^H is turned down, the patentee expects payoff $p^L d - c$. Hence, it is optimal to make an aggressive offer if

$$\lambda \left[(c_A - \widehat{\alpha}_L) - p^H(d - \widehat{\alpha}_D)\right] + (1 - \lambda)\left[p^L d - c\right] \geq (c_A - \widehat{\alpha}_L) - p^L(d - \widehat{\alpha}_D).$$

Note that if litigation costs insurance is higher, then it is more likely that the aggressive offer is optimal. On the other hand, an increase in damages insurance makes an aggressive offer more likely if $p^L - \lambda p^H > 0$, that is, if the fraction of high types is relatively low.

Hence, the patentee makes an aggressive offer provided that λ exceeds the following cutoff:

$$\widehat{\lambda} = \frac{(c + c_A) - \widehat{\alpha}_L - p^L \widehat{\alpha}_D}{(c + c_A) + (p^H - p^L)d - \widehat{\alpha}_L - p^H \widehat{\alpha}_D}.$$

As a benchmark, note that for the case of no insurance ($\alpha = 0$), the cutoff satisfies

$$\widehat{\lambda} = \frac{(c + c_A)}{(c + c_A) + (p^H - p^L)d}.$$

Consider the possibility of a pooling equilibrium. Let's focus on the case where the fraction of high-risk types is not too high, $\lambda < \frac{p^L}{p^H}$, so that insurance is most valuable when damages insurance is set to zero and just litigation costs insurance is sold. Then, the most valuable insurance is such that the patentee is just willing to make a weak offer. This occurs when the true value of λ equals the cutoff above, and holds for

$$\widehat{\alpha}_L = c_A + c - \frac{d\lambda}{1 - \lambda}(p^H - p^L).$$

Note first that this level of insurance is $c_A + c$ when $\lambda = 0$. This exceeds the agent's litigation cost. Imposing the constraint $\widehat{\alpha}_L \leq c_A$, we then solve to find $\widehat{\alpha}_D = \frac{(1-\lambda)c - \lambda d\left(p^H - p^L\right)}{p^L - \lambda p^H}$. When $\lambda = 0$, we find $\widehat{\alpha}_D = \frac{c}{p^L}$, as earlier. The constraint $\widehat{\alpha}_L \leq c_A$ binds for $\lambda \in \left[0, \frac{c}{c + d(p^H - p^L)}\right)$. As λ increases slightly above 0, damages insurance is not as generous, because more generous damages insurance would make an aggressive offer more likely. Similarly, willingness to pay falls with λ. Note that when the offer is weak under no insurance and under insurance, willingness to pay is the same for both types, because types get the same offer.

We now show that this contract may form a pooling equilibrium.

Proposition. *The pooling contract* $(\widehat{\alpha}_L, \widehat{\alpha}_D) = \left(c_A, \frac{(1-\lambda)c - \lambda d\left(p^H - p^L\right)}{p^L - \lambda p^H}\right)$*, sold for a price of zero, is an equilibrium for sufficiently low* λ*.*

The intuition for this result is as follows. If any less generous insurance is offered for a price of zero, neither type of agent will buy it. The low type's payoff is the same under a weak offer, an aggressive offer, and litigation. The high type earns rent under a weak offer, but not in the other cases. If slightly more generous insurance is offered and both types buy this insurance, then the patentee will make an aggressive settlement offer. The high type will have a higher increase in willingness to pay, so anything that attracts the low type will also attract the high type. For any contract that attracts only high types, the high type that purchases that contract will then receive an aggressive offer and earn no rent. So high types will want to deviate only if low types also deviate.

Hence, consider the alternative contract $(\widehat{\alpha}_L, \widehat{\alpha}_D) = \left(c_A, \frac{c}{p^L}\right)$. This maximizes the willingness to pay for the low type. If $\lambda > 0$, this contract induces an aggressive offer. Under an aggressive offer, low types reject the offer, so the insurer expects to pay $(1-\lambda)(c_A+c)$ in claims. The increase in willingness to pay is $c - p^L\left(\frac{(1-\lambda)c-\lambda d\left(p^H-p^L\right)}{p^L-\lambda p^H}\right)$. For small λ, this increase in willingness to pay is not enough to cover the extra cost. Hence, there is no alternative profitable pooling contract.

B.3.2 Reinganum-Wilde

As an alternative model of bargaining, assume that the agent knows p, but the insurer and patentee know only that p is distributed according to increasing density $F(p)$ on [0,1].[18] Everything else, including insurance purchased $\{\widehat{\alpha}_L, \widehat{\alpha}_D\}$, is common knowledge. Let the agent make a take-it-or-leave-it settlement offer S, and let the probability that a settlement offer is accepted, $q(S)$, depend upon the particular offer made. For the agent, the payoff is

$$\Pi_E = q(S)\left[-p(d-\widehat{\alpha}_D) - (c_A - \widehat{\alpha}_L)\right] + \left[1-q(S)\right](-S).$$

For the patentee, the payoff is

$$\Pi_P = q\left[p(S)d - c\right] + [1-q]S.$$

Differentiating Π_P with respect to the rejection policy, we find

$$\frac{d\Pi_P}{dq} = p(S)d - c - S = 0.$$

[18] With private information set up this way, it would probably be more realistic to let the uncertainty be over damages d rather than the probability of liability.

This expression does not depend upon q. Hence, if it is positive, the patentee always rejects. If it is negative, the patentee always accepts. We will seek an interior solution, where $q(S) = 0$, so that there is some probability of rejection between 0 and 1 for any settlement offer that reflects a $q \in (0, 1)$. Hence, we have

$$S = pd - c.$$

The agent chooses its settlement offer S to maximize its profit. We have

$$\frac{d\Pi_E}{dS} = q'(S)\left[-p(d - \widehat{\alpha}_D) - (c_A - \widehat{\alpha}_L) + S\right] - [1 - q(S)].$$

Imposing $S = pd - c$ and rearranging, we then have

$$q'(S)\left[c\left(1 - \frac{\widehat{\alpha}_D}{d}\right) + c_A - \widehat{\alpha}_L - \left(\frac{\widehat{\alpha}_D}{d}\right)S\right] + [1 - q(S)] = 0.$$

This is a first-order differential equation in S, which yields solution

$$q(S) = 1 - \omega\left[c\left(1 - \frac{\widehat{\alpha}_D}{d}\right) + c_A - \widehat{\alpha}_L - \left(\frac{\widehat{\alpha}_D}{d}\right)S\right]^{\left(\frac{d}{\widehat{\alpha}_D}\right)}.$$

Imposing boundary condition $q(\underline{S}) = 0$, where $\underline{S}$ corresponds to $p = 0$, we find

$$q(S) = 1 - \left[\frac{c\left(1 - \frac{\widehat{\alpha}_D}{d}\right) + c_A - \widehat{\alpha}_L - \left(\frac{\widehat{\alpha}_D}{d}\right)S}{c\left(1 - \frac{\widehat{\alpha}_D}{d}\right) + c_A - \widehat{\alpha}_L - \left(\frac{\widehat{\alpha}_D}{d}\right)\underline{S}}\right]^{\left(\frac{d}{\widehat{\alpha}_D}\right)}.$$