Microeconomic foundations of macroeconomic models

Models of intertemporal optimization

We introduced dynamic optimization in the context of the Ramsey model of growth. This optimization model provides a useful framework for thinking about how economic decisions are made when time matters.

In the comparative statics macroeconomics models just covered, we assumed decision rules took specific forms (i.e. had specific arguments) without deriving these decision rules from underlying choice-theoretic framework (i.e. utility maximization). Here we apply the theory of dynamic optimization/choice to these decision rules: optimization of dynamic objective function subject to dynamic constraints. Dynamic optimization implies forward looking behavior.

We will first look at two household decisions: how much of income to consume or save; how much labor to supply. In general, these decisions are determined jointly; but we start with a simpler case where labor supply is exogenous. This allows us to focus on intertemporal consumption choice. Then we generalize, include labor/leisure as choice to focus on labor supply.

As discussed with growth, dynamic optimization generally requires the use of very sophisticated and complex mathematical tools. However, we use a simplified framework without worrying too much about technicalities. For example, we will focus on two-period case, which is amenable to graphical analysis (it turns out that under certain conditions, general problems collapse to a series of two-period problems). The only math background needed is constrained optimization (and difference equations).

As we did with the Ramsey model, we will work in a discrete time framework. There, we focused on the problem of a central planner. Here, we more specifically look at the aggregate consumer’s problem.

Intertemporal substitution of consumption

Assume a representative household with planning horizon $T$ periods (0 is present, the time at which decisions are made). $r$, the real interest rate,
is exogenous (taken as given by the household) and constant, as is present and future income. All variables are in real terms, ignoring the existence of money. **Perfect foresight – uncertainty is ignored.**

Objective function and asset accumulation constraint (the former describes preferences for intertemporal consumption, the latter describes the possibility of intertemporal tradeoffs through purchases of assets):

\[
V_0 = u_0(c_0, c_1, \ldots, c_T) \tag{1}
\]
\[
A_{t+1} = (1 + r)(A_t + y_t - c_t) \quad t = 0, 1, \ldots, T \tag{2}
\]

where \(A_0\) is given. Observe the sense in which this problem is one of intertemporal substitution: consuming less at time \(t\) increases saving and thus future assets; but these assets allow more future consumption. To the extent that hh’s are able to borrow and lend in perfect capital markets, consumption is limited only by initial wealth and lifetime income. Given this concept of total wealth, the hh’s problem is to decide when to consume.

Preferences are given in general form here; we consider the special case:

\[
V_0 = \sum_{t=0}^{T} \beta^t u(c_t) \tag{1'}
\]

where \(\beta\) is personal rate of time preference (it measures the households impatience – e.g. the smaller \(\beta\), the more impatient, since hh “discount” future utility a lot, a lot of future consumption needed to compensate for waiting) and \(u(\cdot)\) is utility function at a point in time. Assume this function is constant.

HH maximizes (1′) subject to (2), with respect to the choice variables – consumption in each period of the planning horizon. This determines the hh’s optimal consumption path. Use the usual constrained optimization methods, realizing that choices are made for each period in the planning horizon. (Ignore end-point considerations; i.e. transversality conditions.)
Lagrangean method:

\[ L = \sum_{t=0}^{T} \beta^t u(c_t) - \sum_{t=0}^{T} \lambda_t [A_{t+1} - (1 + r)(A_t + y_t - c_t)] \]

\[ \frac{\partial L}{\partial c_t} = \beta^t u'(c_t) - \lambda_t (1 + r) = 0, \quad t = 0, \ldots, T \]
\[ \frac{\partial L}{\partial A_{t+1}} = -\lambda_t + \lambda_{t+1} (1 + r) = 0, \quad t = 0, 1, \ldots, T - 1 \]
\[ A_{T+1} = 0 \]

Note that these conditions must hold for each period in planning horizon, along with the accumulation constraint (derivative of Lagrangean wrt multipliers). Last condition is intuitive – consume wealth at the end, since only consumption yields utility. These are necessary (first-order) conditions for optimal time path. Assume sufficient conditions hold.

FOC are called Euler equations. Combine the first two:

\[ \beta u'(c_{t+1})(1 + r) = u'(c_t) \quad t = 0, \ldots, T - 1 \] (3)

This says that a necessary condition for hh’s optimal consumption path is that, for any two consecutive periods, the discounted marginal costs of increasing current consumption (LHS of 3) must equal the marginal benefits (RHS of 3). Marginal cost is the value of future consumption given up – sacrificing one unit of current consumption yields \((1 + r)\) units of future consumption; this yields marginal value of \((1 + r)u'(c_{t+1})\). This marginal value occurs in the future, so it must be discounted back to present to control for time preference; hence the presence of \(\beta\).

Income and wealth will affect consumption path through the accumulation constraint.

Set of Euler equations defines a nonlinear difference equation system. With constraints, a difference equation system is formed. The solution of this system determines optimal time paths, which is just consumption as function of time.
For intuition, let $T = 1$; i.e. consider a two-period model. HH chooses $c_0$, $c_1$ and $A_1$ ($A_0$ is given and $A_2$ is zero as noted above)

$$V = u(c_0) + \beta u(c_1)$$
$$A_1 = (1 + r)(A_0 + y_0 - c_0)$$
$$c_1 = A_1 + y_1$$

Euler equation:

$$\frac{u'(c_0)}{\beta u'(c_1)} = (1 + r)$$

Interpretation:

$$dV = u'(c_0)dc_0 + \beta u'(c_1)dc_1 = 0$$
$$\frac{\partial c_1}{\partial c_0} = -\frac{u'(c_0)}{\beta u'(c_1)}$$

LHS of Euler equation is slope of indifference curve; shows tradeoff of current and future consumption to maintain constant level of utility. The curvature of the indifference curve reflects *consumption smoothing*: the slope of the indifference curve is flatter the higher is $c_0$, and steeper the higher is $c_1$. For example, if the utility function is log linear (i.e. Cobb-Douglas), then

$$\frac{\partial c_1}{\partial c_0} = -\frac{c_1}{\beta c_0}.$$

If $c_1$ is small, then the slope is flat; if $c_0$ is small, then slope is steep. Illustrate on graph.

Combine last two equations of (4)

$$c_1 - y_1 = (1 + r)(A_0 + y_0 - c_0)$$
$$\frac{\partial c_1}{\partial c_0} = -(1 + r)$$

RHS is slope of intertemporal budget line; the real interest rate determines the available tradeoff of consumption over time. The Euler equation is the usual tangency condition: slope of indifference curve equals slope of budget line.
To determine intercepts, use accumulation constraint:

\[ c_1 = y_1 + (1 + r)(A_0 + y_0) - (1 + r)c_0 \]

Movements along budget line reflect intertemporal consumption reallocation, for fixed wealth and income; i.e. borrowing (down to right) and lending (up to left). If \( c_0 = A_0 + y_0 \) and \( c_1 = y_1 \) – neither borrowing or lending. Points to right of this point reflect borrowing on future income; points to left are lending.

Consider the no-borrowing/no-lending point. Slope of indifference curve is flatter than budget line. Reducing \( c_0 \) by one unit can yield more \( c_1 \) than is required to maintain indifference. Thus, reducing \( c_0 \) in favor of \( c_1 \) must increase utility of household. Thus, this point cannot be optimal; should reduce current spending, increase future spending.

Suppose that the interest rate changes; this would be an example of extrinsic dynamics, since we are in effect looking at dynamic multipliers. In particular, suppose household is a net lender and \( r \) rises. Budget line rotates around the no-borrowing/no-lending point (the higher interest rate reduces the present value of future income but increases future value of current income).

Movement along the original indifference curve represents the substitution effect (utility-compensated) – lender increases lending. But there is also an income effect, since the budget line is farther to the right for the lender. The income effect tends to increase both current and future consumption. If \( c_1 \) rises and \( c_0 \) falls, the substitution effect outweighs the income effect with respect to current consumption. Recall that this is consistent with the second argument in the consumption function in the static model.

For a borrower, substitution effect works in same direction, but income effect has a negative impact on current and future consumption (borrower will borrow less).

Changes in income: Assume logarithmic utility function.

\[ u(c) = \ln(c) \]
Euler equation:
\[
\frac{u'(c_0)}{\beta u'(c_1)} = (1 + r)
\]

Constraint (eliminate \(A_1\)):
\[
c_0 + (1 + r)^{-1}c_1 = A_0 + y_0 + (1 + r)^{-1}y_1
\]

(When the constraint is written this way, it says that present value of consumption is limited to current wealth plus present value of future income flows. Note that this step implicitly imposes the transversality condition: \(A_2 = 0\). In effect, this puts the system on its saddlepath.)

With log utility, Euler equation implies
\[
c_1 = \beta(1 + r)c_0;
\]

optimal consumption path is described by first order difference equation. Combining this with the budget constraint (solve budget constraint for \(c_1\) then plug into Euler equation):
\[
c_0 = \frac{1}{\beta(1 + r)}[(1 + r)(A_0 + y_0 - c_0) + y_1]
\]

Solving this for \(c_0\) then using this result in the Euler equation gives optimal time path for consumption as a function of permanent income (i.e. consumption function):
\[
c_0 = \frac{1}{(1 + \beta)}[A_0 + y_0 + (1 + r)^{-1}y_1]
\]
\[
c_1 = \frac{\beta}{(1 + \beta)(1 + r)}[A_0 + y_0 + (1 + r)^{-1}y_1]
\]

It is optimal to consume a fraction of permanent income, which is sum of initial wealth and discounted future income (labor wealth). Only exogenous changes in permanent income affect optimal consumption plans.

Some quantitative experiments: suppose that \(\beta = 0.95\), \(r = 0.06\).
1) increase in current income

\[
\frac{\partial c_0}{\partial y_0} = \frac{1}{1 + \beta} = .51
\]

\[
\frac{\partial c_1}{\partial y_0} = \frac{\beta}{(1 + \beta)(1 + r)} = .52
\]

Consumption is “smoothed” intertemporally; a current shock to income is allocated through time, consumption path is relatively steady. The shape of the indifference curves implies that consumption smoothing is optimal: if current consumption is high, hh willing to give up a lot of current consumption to get just a little future consumption.

2) equal increase in \(y_0\) and \(y_1\): a change in permanent income (\(dy_0 = dy_1\))

\[
\frac{\partial c_0}{\partial y_p} = \frac{1}{(1 + \beta)} + \frac{1}{(1 + \beta)(1 + r)} = .99
\]

\[
\frac{\partial c_1}{\partial y_p} = \frac{\beta}{(1 + \beta)(1 + r)} + \frac{\beta}{1 + \beta} = 1.01
\]

To get these effects, add the coefficients on \(y_0\) and \(y_1\). Both current and future desired consumption rise by more than in case one, since permanent income causes a larger shift to the right in the budget line than before.

3) Transitory change in income: \(dy_0 = -dy_1\)

\[
\frac{\partial c_0}{\partial y_T} = \frac{\partial c_0}{\partial y_0} - \frac{\partial c_0}{\partial y_1} = \frac{1}{1 + \beta} - \frac{1}{1 + \beta} \frac{1}{1 + r} = .03
\]

\[
\frac{\partial c_1}{\partial y_T} = .03
\]

The change in consumption is much smaller since transitory income has only a small impact on budget line (if \(dy_0 = -(1 + r)^{-1}dy_1\), no shift and no change in consumption).

General comments:
1. While this analysis justifies having \(r - \pi\) in earlier consumption function, it suggest possible misspecification since only current income
showed up there. How a bad a misspecification depends on how closely current income is to permanent income. M. Friedman used this permanent income hypothesis to explain why consumption responded less to income over the business cycle than over the long run.

2. We have assumed that income (and interest rate) are known with certainty. Under certain conditions, we can replace future values with expectations (e.g. rational expectations) in the case of uncertainty.

3. HH’s may be liquidity constrained; i.e. they borrow at higher rate than they lend, or they can’t borrow at all. In this case, consumption would be much more sensitive to current income, even transitory income.


Reducing T-period problem to 2-period case:

\[ T = 1 \rightarrow t = 0, 1 \]

\[ L = u(c_0) + \beta u(c_1) - \lambda_0[A_1 - (1+r)(A_0 + y_0 - c_0)] - \lambda_1[A_2 - (1+r)(A_1 + y_1 - c_1)] \]

First order conditions:

\[ \frac{\partial L}{\partial c_0} = u'(c_0) - \lambda_0(1 + r) = 0 \]  
(1)

\[ \frac{\partial L}{\partial c_1} = \beta u'(c_1) - \lambda_1(1 + r) = 0 \]  
(2)

\[ \frac{\partial L}{\partial A_1} = -\lambda_0 + \lambda_1(1 + r) = 0 \]  
(3)

\[ A_2 = 0 \]  
(4)

\[ \frac{\partial L}{\partial \lambda_0} = (1 + r)(A_0 + y_0 - c_0) - A_1 = 0 \]  
(5)

\[ \frac{\partial L}{\partial \lambda_1} = (1 + r)(A_1 + y_1 - c_1) = 0 \]  
(6)

Combine 1,2,3 by eliminating \( \lambda_0, \lambda_1 \):

\[ \frac{\beta u'(c_1)}{u'(c_0)} = (1 + r)^{-1} \]  
(7)
This equation, along with the constraints in 5 and 6 (note that 6 implies that \( c_1 = A_1 + y_1 \)), comprise a 3 equation system that in principle determines the 3 unknowns \( c_0, c_1, \) and \( A_1 \), given the exogenous process \( y_0, y_1 \).

**Intertemporal substitution of labor/leisure**

Generalize household optimization to allow substitution of leisure and work intertemporally. Leisure provides utility, which was ignored in the previous analysis. Cost of leisure is value of consuming produced goods since the alternative to leisure is work, which generates income.

This model dates back to Lucas and Rapping (1969), who were trying to develop a labor supply model that explained salient facts of employment and wages; namely, labor supply is highly inelastic in the long-run but elastic in the short-run (with respect to real wage).

As before, focus on a two-period model.

\[
\begin{align*}
\max \quad & V = u(c_0, n_0) + \beta u(c_1, n_1) \\
\text{s.t.} \quad & c_1 = (1 + r)(w_0n_0 - c_0) + w_1n_1
\end{align*}
\]

where \( n \) is labor supply, so \( u_n < 0 \), \( w \) is real wage (in terms of produced good).

In the exogenous labor model above (where income exogenous), there was only one margin – current consumption versus future consumption. Here, there are 3 important margins: current vs. future cons.; current vs future labor; and consumption vs labor (intratemporal). But these margins are not independent; the decisions are joint.

\[
\Lambda = u(c_0, n_0) + \beta u(c_1, n_1) - \lambda[c_1 - (1 + r)(w_0n_0 - c_0) - w_1n_1]
\]

The household is choosing consumption and labor to maximize utility, taking prices as given. Deriving decision rules. First order conditions:

\[
\frac{\partial \Lambda}{\partial c_0} = u_1(c_0, n_0) - \lambda(1 + r) = 0 \quad (1)
\]
\[ \frac{\partial \Lambda}{\partial c_1} = \beta u_1(c_1, n_1) - \lambda = 0 \quad (2) \]
\[ \frac{\partial \Lambda}{\partial n_0} = u_2(c_0, n_0) + \lambda(1 + r)w_0 = 0 \quad (3) \]
\[ \frac{\partial \Lambda}{\partial n_1} = \beta u_2(c_1, n_1) + \lambda w_1 = 0 \quad (4) \]
\[ \frac{\partial \Lambda}{\partial \lambda} = c_1 - (1 + r)(w_0n_0 - c_0) - w_1n_1 = 0 \quad (5) \]

Margin of intratemporal consumption/labor: combine 1 & 3 or 2 & 4:

\[ -\frac{u_2(c_t, n_t)}{u_1(c_t, n_t)} = w_t \]

Interpretation:

From constraint:

\[ c_0 = w_0n_0 + \frac{w_1}{1 + r}n_1 - \frac{c_1}{1 + r} \]
\[ \frac{\partial c_0}{\partial n_0} = w_0 \]

From utility function:

\[ dV = u_1(c_0, n_0)dc_0 + u_2(c_0, n_0)dn_0 + \beta u_1(c_1, n_1)dc_1 + \beta u_2(c_1, n_1)dn_1 \]
\[ \frac{\partial c_0}{\partial n_0} = -\frac{u_2(c_0, n_0)}{u_1(c_0, n_0)} > 0 \]

Intertemporal consumption – combine 1 and 2 (holding labor path fixed):

\[ \frac{\beta u_1(c_1, n_1)}{u_1(c_0, n_0)} = (1 + r)^{-1} \]

From constraint:

\[ \frac{\partial c_1}{\partial c_0} = -(1 + r) \]
From utility function:
\[
\frac{\partial c_1}{\partial c_0} = -\frac{u_1(c_0, n_0)}{\beta u_1(c_1, n_1)} \]

Intertemporal labor – combine 3 and 4 (holding consumption path fixed):
\[
\frac{\beta u_2(c_1, n_1)}{u_2(c_0, n_0)} = (1 + r)^{-1} \frac{w_1}{w_0}
\]

From constraint:
\[
\frac{\partial n_1}{\partial n_0} = -\frac{w_0}{w_1} (1 + r)
\]

From utility function:
\[
\frac{\partial n_1}{\partial n_0} = -\frac{u_2(c_0, n_0)}{\beta u_2(c_1, n_1)}
\]

At point A: increase future labor by one unit; to remain indifferent (holding c) fixed, current labor needs to be reduced to B only; however, by moving along budget line (given relative wages and r), hh can supply even less current labor (point C) and still maintain present value of income. This makes hh better off.

Alternative interpretation: rewrite FOC as
\[
u_2(c_0, n_0) = \beta u_2(c_1, n_1)(1 + r) w_0 \frac{1}{w_1}
\]

Marginal cost of working one current hour = marginal value of reducing future labor, which is discounted future marginal disutility of labor times reduction in future labor that allows consumption path to remain fixed.

Permanent change in real wage: \( dw_0 = dw_1 > 0 \) Look at \((c_0, c_1)\) and \((n_0, n_1)\) figures. Since consumption/labor decision is joint, the effect of a permanent change in wage on labor supply depends on what happens to consumption. An increase in \(w_0\) and \(w_1\) will shift consumption budget line to the right (tending to increase current and future consumption for fixed labor path) and will shift labor budget line to the left (tending to decrease labor for
fixed consumption path). These shifts represent substitution effects (at any time, leisure is more expensive so reduce leisure) and income effects (at any time, less labor is needed to generate a given consumption path). Since there is no change in the intertemporal tradeoff between labor over time, these effects will tend to cancel, so that there will be only small changes in labor supply. For some specification of utility, these effects will cancel, so that labor path is unaffected, but consumption path increases (consumption budget line shifts out for higher wages but no change in labor).

Temporary change in real wage: \( dw_0 > 0; dw_1 = 0 \) In this case, change in current wage will alter intertemporal tradeoff; the slope of budget lines will change. In effect, the temporarily higher current wage means that current leisure is more expensive than future leisure. (For a permanent change in wage, there was no change in tradeoff between leisure over time.) Cet. par. hh’s will tend to substitute away from current leisure for future leisure; i.e. work more now, less later. This leads to elastic labor supply.

Another way to view this – look at third Euler equation relating to labor path. If current and future wages change proportionately (a permanent change), then RHS of the equation doesn’t change, so less likely for employment to change. However, if \( w_0 \) rises, with \( w_1 \) fixed, RHS falls so that \( n_0 \) must rise (which increases the denominator on the LHS, because of increasing marginal costs of working), \( n_1 \) must fall.

*Government finance and Ricardian Equivalence*

Under ideal circumstances, intertemporal choice implies that the means by which the government sector finances its deficit (holding the level of its spending constant) has no effect on the economy. Recall that reducing lump sum taxes in the static models affected the interest rate; this result is not necessarily obtained when forward looking behavior is considered.

We will look at this issue in the simplest possible way – 2 period model. We now introduce a government sector, which can consume (but not produce), collect lump-sum taxes, and borrow by issuing debt (interest paying bonds, not money).
Since government spending (i.e. consumption) does not affect HH’s utility, the only effect it can have on HH’s is on it’s budget constraint.

Household sector’s lifetime budget constraint (same as before, except for payment of taxes), assuming \( r \) is exogenous and constant:

\[
\begin{align*}
c_0 &= y_0 - s - T_0 \\
c_1 &= y_1 + (1 + r)s - T_1
\end{align*}
\]

which can written compactly (by eliminating saving) as

\[
c_0 + \frac{c_1}{1 + r} = [y_0 + \frac{y_1}{1 + r}] - [T_0 + \frac{T_1}{1 + r}].
\]

Present value of consumption cannot exceed present value of (exogenous) income less present value of taxes paid to gov’t. (We are assuming that hh’s can save only by lending to government, i.e. buying govt debt. Adding other storage technology, e.g. capital, doesn’t change main results.)

The government sector, as long as it can’t borrow forever (i.e. it must pay back its debt at some point; \( b \) cannot approach \( \infty \)) also has an intertemporal budget constraint:

\[
\begin{align*}
g_0 &= T_0 + b \\
g_1 &= T_1 - (1 + r)b
\end{align*}
\]

\( b \) is the flow of bonds issued to finance the deficit in time 0. (Since this is only a 2-period problem, the flow of debt equals the stock of debt.) Note that if the gov’t runs a deficit in period 0, it must run a surplus in period 1 – paying back principal and interest.

\[
g_0 + \frac{g_1}{1 + r} = [T_0 + \frac{T_1}{1 + r}].
\]

Gov’t sector is constrained such that present value of spending equals present value of all tax flows.

Consider the following experiment: holding lifetime gov’t spending fixed, decrease \( T_0 \) (e.g. the gov’t passes a tax rebate by increasing its deficit and
borrowing). Since \( dg_0 = dg_1 = 0 \), \( dT_0 = -\frac{dT_1}{1+r} \) – decrease in taxes today must be matched by an increase in future taxes (equal to present value of the current tax decrease). The increase in the deficit is simply an increase in future tax liabilities.

Notice that this policy has no effect on HH’s budget constraint – in effect, HH’s increase saving in the face of the tax rebate \( (ds = -dT_0) \). They do this to be able to pay for the future tax increase. This is known as Ricardian Equivalence of tax and bond finance.

As \( T_0 \) falls and \( T_1 \) rises, the gov’t NB/NL point moves along its budget line. But at the same time, the HH’s respond by altering their NB/NL; nothing affects their optimal consumption path, since their constraint is unaffected. If government spending changes, in general HH’s budget line shifts, so that consumption is affected. Only if \( dg_0 = \frac{dg_1}{1+r} \) will there be no effect (since the present value of tax revenues need not change). Remember that this is a partial equilibrium (price-taking behavior) result. In general equilibrium, \( r \) will change.

Crucial assumptions: 1) infinite horizons (i.e. hh’s care about future periods); this matters if gov’t lasts beyond hh lifetimes. 2) perfect capital markets (no liquidity constraints). 3) Non-distortionary taxation. These conditions likely not met in practice, but the model illustrates valid reasons for why deficits might matter.

If we allow money finance, we can interpret \( T \) as an “inflation tax”. See Sargent and Wallace on “Unpleasant Monetarist Arithmetic”

Current account in an open economy

Here, we look at a country’s current account (basically its trade balance) in light of intertemporal optimization. At the most fundamental level, current account imbalances allow countries to allocate consumption over time by borrowing and lending with other countries.

Consider a small open economy that can freely trade goods and assets internationally. No production, only income endowments in each of two periods.
Since no government and no firms (no investment), domestic borrowing and lending must be undertaken with other countries.

Defining the current account

The current account balance of a domestic country is the change in the value of its net foreign assets: the flow of lending from the domestic country to foreign countries (if the current account balance is negative, then the domestic country is a net borrower). From the circular flow, we know that

\[ I = S + (T - G) - CA. \]

But since \( I = T = G = 0, S = CA \); i.e. the change in net foreign assets is the difference between total national income and consumption (since there is no investment) – saving. Total national income is the sum of the domestic endowment plus claims on foreign income. The sum is GNP; the first component is GDP. Formally (for any given time period),

\[ CA = y + rB - c \]

where \( CA \) stands for current account balance, \( y \) is gdp (which is an endowment in this model, no production), \( r \) is the interest rate, \( B \) is the current stock of foreign assets held by domestic citizens (so that \( rB \) is interest income from abroad), and \( c \) is domestic consumption. \((y + rB = GNP)\). The logic behind this identity is that domestic citizens increase their holding of foreign assets (their lending to foreigners) when their income exceeds their consumption (note that this would change if domestic firms borrowed to finance investment).

Two-period model – time 0 and time 1. Define the current account balance for each period, assuming that foreign assets are initially zero:

\[ CA_0 = y_0 - c_0 \]
\[ CA_1 = y_1 + rCA_0 - c_1 \]

Since there are no foreign assets initially, \( CA_0 \) is the saving in the domestic economy, which is the accumulation of foreign assets. In period 1, domestic households are endowed with \( y_1 \), plus they earn interest on the accumulated
stock of foreign assets. Since this is a two-period model, make one more restriction: that no wealth is left over (or conversely no debt is left unpaid) in the second period. That is, \( c_1 \) equals total income in the second period plus the value of assets held (or minus debt owed):

\[
c_1 = y_1 + rCA_0 + CA_0.
\]

This implies that

\[
CA_1 = y_1 + rCA_0 - (y_1 + rCA_0 + CA_0)
\]

\[
CA_1 = -CA_0
\]

Plug this into the second equation above, then combine both equations by eliminating \( CA_0 \) yields

\[
c_0 + \frac{c_1}{1 + r} = y_0 + \frac{y_1}{1 + r}
\]

the usual present value constraint. To reiterate, if \( y_0 > c_0, CA_0 > 0 \) domestic households save by exporting produced goods to foreigners in exchange for assets, which are promises to pay future goods produced in foreign country. Thus, a surplus today is followed by a deficit tomorrow, as households spend more than their income in the future period. Likewise, a deficit today would be backed up by a surplus tomorrow, so that households can pay back debt.

Consider no trade/no borrowing or lending in world-wide financial markets. This point would be optimal if \( r = r_A \). However if \( r = r_w < r_A \), then, given preferences and endowments, domestic country has a “comparative advantage” in borrowing (and paying back later), while rest of world has comparative advantage in lending (and consuming more later). Note that domestic country will run a current account deficit in period 0, and a surplus (to pay off debt) in period 1. This gets the country to a higher utility level. The second figure shows the opposite case where domestic country is lender in period 0 (current account surplus).

The point of all this is that trade deficits are fundamentally motivated by intertemporal trade; there is no a priori reason, for example, to think that trade deficits are “bad” for the economy.
In these examples, comparative advantage is due to differences in time-preference, which motivates borrowing and lending given arbitrary endowments. More generally, a domestic economy will run a trade deficit when the rate of return from investing in that country is higher than that in the rest of the world (assuming perfect capital mobility). The domestic country then has a comparative advantage in accumulating capital, so they borrow from the rest of the world to invest, then pay back in the future. This is efficient. Production is on the world wide production frontier.

*Investment*

Now use the tools of intertemporal optimization to derive firm’s optimal investment decision. This treatment follows Romer’s closely. Consider the behavior of one firm in an industry of $n$ firms. Assume only one type of good, which can be either consumed or invested (in which case is permanently becomes a capital good). This implies that there is a perfectly elastic supply of capital at a constant relative price of one. Each firm is identical, and faces a cost of adjusting its capital stock. That is, resources are used up to make capital useful for production. This cost is increasing in the rate at which the capital stock changes. Net revenues (net of costs of other inputs) is proportional to the individual firm’s stock of capital, but diminishing in the industry’s stock of capital. The latter assumes an industry-wide positive externality, but that the industry faces a downward sloping demand curve for its output. Firms take the industry level of capital as given; the same for initial individual capital.

Definitions: $K_t$ – industry capital at time $t$; $k_t$ – firm capital stock, $I_t = k_{t+1} - k_t$ – firm’s net investment; $\delta$ – rate of physical depreciation; $\pi(K)$ – net revenues per unit of firm’s stock of capital given other inputs, where $\pi'(K) < 0$; $C(I)$ is the adjustment cost function, where $C''(I) > 0$, $C''(I) > 0$ and $C(0) = 0$

The firm is assumed to maximize its value; i.e. maximize present value of its profits (cash flow or dividends). At any point in time, its profits are

$$\Pi_t = \pi(K_t)k_t - I_t - \delta k_t - C(I_t).$$

Assume that the firm’s marginal opportunity cost of capital is simply the real interest rate, $r$, and that this interest rate is constant. (If $r$ is the
yield on bonds, then we are assuming that equities and bonds are perfect substitutes.) If the firm has an infinite horizon, then we have the following dynamic optimization problem:

\[
\max_{t} \sum_{t=0}^{\infty} (1 + r)^{-t} \Pi_t
\]

subject to \( k_{t+1} = k_t + I_t \),
the Lagrangean of which is

\[
\Lambda = \sum_{t=0}^{\infty} (1 + r)^{-t} [\pi(K_t)k_t - I_t - \delta k_t - C(I_t) - q_t(k_{t+1} - k_t - I_t)],
\]

where \( q_t = (1 + r)\lambda_t \) and \( \lambda_t \) is the usual Lagrange multiplier. Since \( q \) is a multiplier, it has the usual interpretation of the marginal value of relaxing the constraint; in this case, it is the increased future value of the firm due to a one unit increase in the stock of capital. Remember that the actual price of capital is one; \( q \) can be interpreted as a shadow price, or willingness to pay for capital.

The firm chooses the entire path of \( I_t \) and \( k_{t+1} \) to maximize its objective subject to the constraint. We have the following first order conditions (we will assume second order conditions hold and will ignore the transversality condition):

\[
\frac{\partial \Lambda}{\partial I_t} = (1 + r)^{-t}[-1 - C'(I_t) + q_t] = 0
\]

\[
\rightarrow q_t = 1 + C'(I_t)
\]

\[
\frac{\partial \Lambda}{\partial k_{t+1}} = -(1 + r)^{-t}q_t + (1 + r)^{-t+1}[\pi(K_{t+1}) - \delta + q_{t+1}] = 0
\]

\[
\rightarrow \pi(K_{t+1}) = rq_{t+1} - (q_{t+1} - q_t) - r(q_{t+1} - q_t) + \delta
\]

Interpretation of Euler equations: The first says that the firm will invest until the marginal benefit of investing \( (q_t) \) equals the marginal cost of investing (the price of a unit of capital \( -1 \) – and the resources used up to adjust the stock of capital). One unit of investment reduces current profits,
but leads to higher firm value through more capital and future profits. Also, note that we can solve this equation for investment:

\[ I_t = C^{\prime \prime -1}(q_t - 1). \]

This has an interpretation as an investment demand function.

To interpret the second condition, rewrite:

\[ q_t = (1 + r)^{-1}[\pi(K_{t+1}) - \delta + q_{t+1}]. \]

This says that the marginal cost of buying a unit of capital (its shadow value) equals its discounted future benefits: the marginal revenue product of capital (less the rate at which capital depreciates, which can be thought of as resources used to maintain a unit of capital) plus the resale shadow value.

Here is an alternative interpretation. Manipulate the second Euler equation as follows:

\[
(1 + r)q_t = \pi(K_{t+1}) - \delta + q_{t+1} \\
\pi(K_{t+1}) = \left( r + \frac{\delta}{q_t} - \frac{q_{t+1} - q_t}{q_t} \right) q_t \\
\frac{\pi(K_{t+1})}{q_t} = \left( r + \frac{\delta}{q_t} - \frac{q_{t+1} - q_t}{q_t} \right). 
\]

The left-hand-side is the marginal return of increasing the capital stock by one unit next period (relative to the current “price” of capital); the RHS is the marginal user cost of increasing the capital stock: \( r \) is the cost of financing the capital, the second term is the amount of capital lost in production relative to price, and the last term is the rate at which the value of capital appreciates over the period. Note that when the value of capital rises, the user cost is reduced. The Euler equation states that the rate of return to holding capital (adding one unit of future capital equals the marginal opportunity (user) cost.

Note that the Euler equations describe a two-variable difference equation system in \( K_t \) and \( q_t \). (The first is actually in terms of a single firm, but we
simply need to scale by \( n \) to make it describe aggregate capital. The first shows \( K_{t+1} \) as a function of \( K_t \) and \( q_t \) and describes the dynamic behavior of capital; the second shows \( q_{t+1} \) as a function of \( q_t \) and \( K_{t+1} \) and describes the dynamics of the shadow price. Let’s use a phase diagram to analyze the dynamics of the model.

To simplify, assume that \( C(I_t) = I_t^2 \). Then the first equation implies that \( I_t = K_{t+1} - K_t = .5(q_t - 1) \). The capital stock is thus stationary when \( q_t = 1 \). Furthermore, when \( q_t > 1 \rightarrow I_t > 0 \) so that capital is growing, and when \( q_t < 1 \rightarrow I_t < 0 \) so that capital is shrinking. Show on phase diagram.

From the second equation, it is evident that \( q \) is stationary when

\[
q_{t+1} = \frac{\pi(K_{t+1}) - \delta}{r}.
\]

Since \( \pi'() \) is negative, the locus of points such that \( q_{t+1} = q_t \) is downward sloping. See phase diagram. To understand the dynamics of \( q \) off this locus, look at full equation above. As \( K \) increases holding \( q_t \) constant, the right-hand-side of the equation will fall since \( \pi \) is falling in \( K \). Therefore, \( q_{t+1} \) will have to rise above \( q_t \). The rise in \( K \) will reduce the marginal benefit of adding capital, so the shadow price in the future must rise to compensate.

Alternatively, suppose that for a given value of \( q \), \( K \) is to the right of the locus. Then the rate of return from capital is lower than before. For fixed \( r \) and \( \delta \), this means that the rate of return of capital will be lower than user cost, unless the value of capital appreciates over the period. Thus, the imputed price of capital (\( q \)) must rise over time to compensate the firm for the reduced return; if the capital price does not appreciate, then the additional capital will not be willingly held. In other words, to get the firm to willingly hold the less valuable extra unit of capital, the current value of capital must be lower than its future value.

An aside: note from the equation above that when \( q \) is stationary, \( q_t - 1 = \frac{\pi(K_t) - (r + \delta)}{r} \). This shows that our investment equation above is similar in form to the one we assumed in our comparative statics models. Here, we have derived this function explicitly from micro foundations.
This model of investment is called the q-theory of investment, after work by Tobin. The shadow price \( q \) has an interpretation as the ratio of the market value of capital to the replacement cost of capital (its price, which in this case is trivially equal to one). The theory says that firms will invest whenever the market value of capital exceeds the replacement price. This theory is also consistent with the idea that firms will invest when the rate of return of capital exceeds the firms opportunity cost. The opportunity cost of capital is the discount rate used to price the firm’s equity (i.e. to discount future profits of using capital); the rate of return equates the replacement price of capital to discounted profit flows. If the market price exceeds the replacement price, then it must be the case that the opportunity cost is lower than the rate of return of capital.

*Dynamic analysis using the phase diagram*

It can be shown that the model implies a saddlepath, along which \( q \) and \( K \) will travel over time to reach the steady-state, in which \( q \) and \( K \) are constant.
Comparative dynamics

1) Increase in output (demand for industry output): increase in $\dot{q} = 0$ locus; $q$ rises, inducing investment and growth in capital stock.

2) Increase in interest rate (opportunity cost of capital): shift $\dot{q}$ locus down; $q$ falls and investment falls; capital stock decreases.

3) Investment tax credit:

$$\Pi_t = \pi(K_t) k_t - I_t + \theta I_t - \delta k_t - C(I_t)$$

$$q_t + \theta = 1 + C''(I_t)$$

$$\dot{K} = 0 \rightarrow q_t + \theta = 1$$
Theories of nominal and real rigidities

We’ve seen that market imperfections can lead to a role for aggregate demand in determining macro activity. In a sense, though, these rigidities were *ad hoc* – they were assumed because they work. However, we should also try to understand why these rigidities might exist. For example, one assumption was that the nominal wage was exogenous. But workers have an incentive to alter their wages in the face of shocks (since it cause a disequilibrium, there are opportunities for everyone to gain). Why wouldn’t they, why won’t the market clear. The same can be said for sticky-price models. In this section, we consider some economic theories of rigidities.

Note: this coverage is very incomplete, and should be seen as only providing a feel for how economists think about some of these issues.

Lucas’s monetary theory of the business cycle

Lucas’s model – based on confusion between source of shocks to the economy; real or nominal. Think of firms separated by islands; each firm determines profit maximizing output (i.e. has a supply curve). Supply depends on relative price – ratio of own market price to aggregate price level (while firms supply output only on their island, they buy from all islands). But there is an informational rigidity (hence the island metaphor)– suppliers can’t immediately observe prices on other islands, thus, they have to guess (form an expectation) of the aggregate price level.

Derivation of aggregate supply:

\[ y_t(z) = \gamma\left[ p_t(z) - E[p_t|I_t(z)] \right] \]

where \( z \) indexes specific markets (islands), \( p_t \) is aggregate price level and \( I_t(z) \) contains local price (\( p_t(z) \) and \( \Omega_{t-1} \)) and all past information. All variables are in logs, so this is a natural rate supply function: only relative prices matter for production decisions. Rational expectations are assumed, conditional on market specific information.

Suppliers come to work and observe own market price. The signal extraction problem is to interpret the change in market price: is it due to an
aggregate shock, in which case the relative price change is zero (i.e. \( p_t(z) \) and \( E[p_t|I_t(z)] \) change by the same amount), or is it due to a relative price change, in which case the argument in the supply curve changes? To the extent that they think it is relative, they will respond by altering output. In general, since producers have only imperfect information, any given change in price will partially be attributed to aggregate shock, partially to relative shock.

More detail on the signal extraction problem:

\[
p_t = E(p_t|\Omega_{t-1}) + \epsilon_t
\]
\[
p_t(z) = p_t + z_t
\]

where \( E\epsilon_t = E z_t = 0 \), \( E\epsilon_t^2 = \sigma^2 \) and \( E z_t^2 = \tau^2 \). Furthermore, \( z \) and \( \epsilon \) are uncorrelated at all lags. These two equations define the aggregate price level and the relative price level: \( \epsilon \) is an aggregate shock affecting all markets, and \( z_t \) is a relative shock affecting only market \( z \). They essentially define relative demand.

Upon beginning work, producers observe \( p_t(z) \), but neither \( \epsilon_t \) (therefore \( p_t \) is not observable at the time production decision is made) nor \( z_t \). To determine supply, producer must make a rational guess about \( p_t \). This is the signal extraction problem: extract the signal (\( p_t \)) from the observation (\( p_t(z) \)).

\[
p_t(z) = E(p_t|\Omega_{t-1}) + \epsilon_t + z_t
\]
\[
E[p_t|I_t(z)] = E(p_t|\Omega_{t-1}) + E[\epsilon_t|p_t(z) - E(p_t|\Omega_{t-1})]
\]
\[
E[\epsilon_t|(\epsilon_t + z_t)] = (1 - \theta)(\epsilon_t + z_t)
\]
\[
1 - \theta = \frac{cov(\epsilon_t, \epsilon_t + z_t)}{var(\epsilon_t + z_t)} = \frac{\sigma^2}{\sigma^2 + \tau^2}
\]

The second equation above uses the recursive projection formula (Sargent p. 226 ff). The third and fourth use well known statistical results (assuming normality). Thus, optimal conditional expectation of price level is

\[
E[p_t|I_t(z)] = \theta E(p_t|\Omega_{t-1}) + (1 - \theta)p_t(z),
\]
a linear combination of the two pieces of information. If variance of aggregate shocks gets large relative to variance of relative shocks, more weight is put on the current observation of \( p_t(z) \): since aggregate shocks are more likely to occur, it is more likely that an observed market specific price change is due to aggregate rather than real shock.

Use this result to get market specific supply, then average across markets to get aggregate supply:

\[
y_t(z) = \gamma \theta [p_t(z) - E(p_t|\Omega_{t-1})]
\]
\[
y_t = \gamma \theta [p_t - E(p_t|\Omega_{t-1})]
\]
\[
y_t = \gamma \left( \frac{\tau^2}{\tau^2 + \sigma^2} \right) (p_t - E_{t-1}p_t)
\]

Implications: As \( \sigma^2 \) increases, output becomes less responsive to changes in aggregate price level. As \( \tau^2 \) increases, output becomes more responsive to changes in price level (i.e. output-inflation tradeoff increases). Since all firms are identical (except for information), the slope of aggregate supply will be flat when aggregate shocks are unlikely. Thus, AD (money) shocks will have real effects when they are unanticipated and when such shocks are perceived to be unlikely. The slope of AS curve depends on relative variances of shocks.

Aggregate demand and equilibrium:

\[
y_t = x_t - p_t
\]
\[
x_t = x_{t-1} + u_t
\]
\[
u_t \sim N(\delta, \sigma_x^2)
\]

This is a very simplified version of aggregate demand. To solve model, equate supply and demand:

\[
x_t - p_t = \gamma \theta (p_t - E_{t-1}p_t)
\]

Solve this equation for \( p_t \), which is a function of expected price. Take expectations on both sides, solve for \( E_{t-1}p_t \) and substitute process for \( x \) to get

\[
E_{t-1}p_t = x_{t-1} + \delta
\]
Using this result in “quasi” reduced form for $p_t$, then this into aggregate demand or supply gives reduced forms for $p_t$ and $y_t$:

\[
\begin{align*}
p_t &= \gamma \theta \delta + \frac{1}{1 + \gamma \theta} x_t + \frac{\gamma \theta}{1 + \gamma \theta} x_{t-1} \\
y_t &= -\frac{\gamma \theta \delta}{1 + \gamma \theta} + \frac{\gamma \theta}{1 + \gamma \theta} u_t
\end{align*}
\]

Aside: Note that $\sigma^2 = (1 + \gamma \theta)^{-2} \sigma_x^2$. That the subjective variance depends on actual processes generating the variables is what is meant by Rational Expectations.

The interesting empirical question is the response of output to changes in demand:

\[
\pi \equiv \frac{\partial y_t}{\partial u_t} = \frac{\gamma \tau^2}{(1 - \pi)^2 \sigma_x^2 + \tau^2 (1 + \gamma)}
\]

where the definitions of $\theta$ and $\sigma^2$ have been used. Although we can’t solve for $\pi$ in closed-form, it is straightforward to show that $\pi$ and $\sigma_x^2$ are inversely related. This gives the main empirical implication of the theory: The response of output to aggregate demand shocks is negatively related to aggregate demand variance.

The intuition is clear: if $\sigma_x^2$ is large, agents are less likely to be fooled by aggregate shocks, so they know not to respond. Lucas’ (1973) cross-country study broadly confirms the hypothesis (see also Kormendi and Meguire, JPE 1984). However, recent evidence can be interpreted in other ways.

26
A simple efficiency wage model

The key idea is that wages are not only a cost but can be a benefit to firms. For example, the higher the wage, the more likely workers will work rather than shirk (with costly monitoring). In such a case, the firm chooses both labor quantity and optimal wage, even in a competitive market.

Assume \( n \) competitive firms that maximize profits by producing and selling output. Labor is the only input. Production technology depends on effective labor; the effectiveness of each worker is positively related to the real wage:

\[
\pi = y - wL \\
y = f[e(w)L]
\]

where \( e(\cdot) \) is effort, \( w \) is real wage, \( e' > 0, e'' < 0 \). The representative firm’s objective is to maximize

\[
\pi = f[e(w)L] - wL
\]

by choosing \( L \) and \( w \). Even though firms are perfectly competitive, they don’t take the wage as given, since it affects productivity. In effect, the real wage is another input into the production process (since it directly affects effort). First-order conditions:

\[
\frac{\partial \pi}{\partial w} = f'(\cdot)Le'(w) - L = 0 \\
\rightarrow f'(\cdot)e'(w) = 1 \\
\frac{\partial \pi}{\partial L} = f'(\cdot)e(w) - w = 0
\]

The first condition says to set wage (given labor quantity) so that marginal revenues of changing the real wage equal the marginal cost. The second condition says to choose quantity of labor (given wage) such that marginal product of additional labor unit equals marginal cost. Combining by eliminating \( f'(\cdot) \):

\[
\frac{e'(w)}{e(w)}w = 1
\]
The wage rate, \( w^* \) that satisfies this condition is called the *efficiency wage*: it is the wage such that the elasticity of effort with respect to the wage is unitary. Given this wage, the quantity of labor hired \( (L^*) \) is determined by second FOC. Since all firms are identical, each firm chooses the same wage and labor quantity. And, since wage and labor are chosen independently of supply, it is possible to have “equilibrium” unemployment.

*Imperfect competition and price setting*

*Limited participation models of the liquidity effect*