1. The model below adds a flat rate income tax to a simple linear macroeconomic model where the price level is exogenous, and requires that the government deficit be financed by money creation. The model is

\[ y_t = -a_1 r_t + g_t \]
\[ m_t = b_1 y_t - b_2 r_t \]
\[ \tau_t = \mu_0 + \mu_1 y_t \]
\[ m_{t+1} = m_t + g_t - \tau_t, \]

where \( y \) is output, \( r \) is the nominal interest rate, \( g \) is government spending, \( m \) is the nominal money stock, \( \tau \) are tax revenues collected by the government, and all coefficients are positive. The price level \( p \) is exogenous by assumption and is therefore ignored in this problem. The first equation is a simplified aggregate demand relationship (the “IS” curve), the second shows the money market equilibrium condition, the third defines the income tax (\( \mu_0 \) is a lump sum amount, while \( \mu_1 \) is the marginal income tax rate), and the fourth requires the Fed to issue money (next period) if there is a budget deficit (there are no bonds). Assume that \( y_t, r_t, \tau_t, m_{t+1} \) are the endogenous variables. The money stock is exogenous at a point in time, but endogenous over time.

a. Find the dynamic multiplier \( \frac{\partial \tau_t}{\partial g_t} \), for all \( k \).

b. What happens to tax revenues in the long-run if the government increases \( g \) once and for all (i.e. permanently)? Explain.

c. What happens to tax revenues in the long-run if lump-sum taxes (\( \mu_0 \)) are increased once and for all (assuming \( g \) is exogenous)? Explain.

d. Suppose we want to “endogenize” the price level? How would you alter the model to allow the price level to be endogenous? (You don’t need to solve this model; just indicate how you would set it up.)

2. Look back at the rational expectations model of the Phillips curve, discussed in class (pp. 32-33). In writing the reduced form for equilibrium output, we assumed that the current demand and supply shocks (\( \epsilon^d_t \) and \( \epsilon^s_t \)) were not included in the information set used to determine expectations. Suppose now that agents observe \( \epsilon^d_t \) when they form their expectations (i.e. the current demand shock is part of the information set \( I_{t-1} \)). What will the reduced form for output look like in this case? Explain the economics.

3. The following problem is a bit more difficult than what you might expect on the mid-term, but it is good practice for understanding unstable roots and forward solutions. Consider Cagan’s model of the price level, which is familiar from class:

\[ m_t - p_t = \alpha (E_t p_{t+1} - p_t) \]

where \( m \) is the natural log of nominal money, \( p \) is the log of the price level, and \( E_t \) denotes the rational expectation operator conditional on information up to and including time \( t \). Assume that \( \alpha < 0 \), and defining \( \beta = \frac{\alpha}{\pi} \), assume that \( \lim_{k \to \infty} \beta^k E_t p_{t+k} = 0 \). Finally, suppose that \( m \) is exogenous and is determined by the following process, where \( \gamma \) is positive but less than or equal to one and \( \epsilon \) is a zero-mean random variable:

\[ m_t = \gamma m_{t-1} + \epsilon_t. \]
a. Note that the second equation is a first-order difference equation in \( m_t \). Keeping in mind that the expected value of \( \epsilon \) is zero, prove that \( E_t m_{t+1} = \gamma m_t \).

b. We know from class that Cagan’s model implies an unstable difference equation for the price level, and we provided a ‘forward’ solution to that difference equation. Use the result of part (a) in this solution to show how the equilibrium price level \( p_t \) depends on \( m_t \).

c. Now, using your knowledge of difference equations, go back to the money difference equation and solve for \( m_t \) as a function of current and past \( \epsilon_t \). Use this result in part (b). If you’ve done this correctly, you will have an expression for the equilibrium price at time \( t \) as a function of current and past \( \epsilon_s \).

d. Discuss the implied dynamics of this model; in particular, what are the dynamic multipliers? Can you come up with an explanation of the economics underlying these multipliers?

e. How does your answer to (d) change if \( \gamma = 1 \)?
1. Income tax and the money finance of government deficits
   a. To find the dynamic multipliers, solve for the model for the equilibrium response of taxes.
      To do this, you might note that the system is block recursive in $y$ and $r$, as well as $y$, $r$
      and $\tau$: $m_{t+1}$ doesn’t show up in the first three equations, so these equations can be solved
      independently of the government’s finance constraint.

      \[
      \begin{pmatrix}
      1 & a_1 & 0 \\
      b_1 & -b_2 & 0 \\
      -b_1 & 0 & 1
      \end{pmatrix}
      \begin{pmatrix}
      y_t \\
      r_t \\
      \tau_t
      \end{pmatrix}
      =
      \begin{pmatrix}
      g_t \\
      m_t \\
      \mu_0
      \end{pmatrix}
      \]

      Using Cramer’s rule, solve for $\tau$ as a function of $g$ and $m$, then use in the last equation to
      get a difference equation in $m$:

      \[
      \tau_t = \mu_0 + \left( \frac{a_1 \mu_1}{b_2 + b_1 a_1} \right) m_t + \left( \frac{\mu_1 b_2}{b_2 + b_1 a_1} \right) g_t
      \]

      \[
      m_{t+1} = -\mu_0 + (1 - \alpha_1) m_t + (1 - \alpha_2) g_t
      \]

      It is reasonable to assume that $0 < 1 - \alpha_1 < 1$. Then the solution to this is straightforward:

      \[
      m_t = -\frac{\mu_0}{\alpha_1} + (1 - \alpha_2) \sum_{i=0}^{\infty} (1 - \alpha_1)^i g_{t-i}
      \]

      Substitute this result into reduced form for $\tau_t$:

      \[
      \tau_t = \mu_0 + \alpha_1 \left[ -\frac{\mu_0}{\alpha_1} + (1 - \alpha_2) \sum_{i=0}^{\infty} (1 - \alpha_1)^i g_{t-i} \right] + \alpha_2 g_t
      \]

      \[
      = \alpha_1 (1 - \alpha_2) \left[ g_{t-1} + (1 - \alpha_1) g_{t-2} + (1 - \alpha_1)^2 g_{t-3} + \cdots \right] + \alpha_2 g_t
      \]

      Notice that $\mu_0$ cancels out. The multipliers are:

      \[
      \frac{\partial \tau_t}{\partial g_t} = \alpha_2
      \]

      \[
      \frac{\partial \tau_t}{\partial g_{t-k}} = \alpha_1 (1 - \alpha_2) (1 - \alpha_1)^{k-1}
      \]

   b. Summing up these multipliers gives the dynamic impact on $\tau_{t+k}$ of a permanent change
      in $g$:

      \[
      \lim_{k \to \infty} \sum_{i=0}^{k} \frac{\partial \tau_t}{\partial g_{t-i}} = \alpha_2 + \alpha_1 (1 - \alpha_2) \sum_{i=1}^{\infty} (1 - \alpha_1)^{i-1} = 1.
      \]

      In the limit as $k \to \infty$, this multiplier approaches 1. Consider the increase in $g$ initially
      – this causes $y$ to rise, but by less than dollar for dollar with $g$ (this can be verified by
      computing $\frac{\partial y}{\partial g}$). This leads to an even smaller rise in tax revenues, assuming $\mu_1 < 1$.
      Thus, the initial effect of an increase in $g$ is to increase the deficit; money therefore begins
      to grow to finance the deficit. $g$ is permanently higher, but the increased money supply
      provides a larger boost to output; therefore, taxes begin to catch up to the increased
      spending. In the long-run, the money supply will converge to a new, higher, constant

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level, so there can’t be a deficit in the long-run. Taxes therefore full catch up with the increase in \( g \) through the increase in output.

c. Since \( \mu_0 \) does not show up in the solution for taxes, an increase in \( \mu_0 \) has no effect on tax revenues. Here’s why: since the difference equation in \( m \) is stable (or stationary), in the steady-state, money can’t change; i.e. \( m_{t+1} = m_t \). Thus, in the steady-state, the government’s budget must be in balance. This comes about because, in the short run, the incipient rise in tax revenues (holding \( g \) fixed) reduces the stock of money, which reduces output and thus taxes (gradually). Therefore, the stock of money will ultimately decrease to a lower steady-state value, causing output to permanently fall, and tax revenues to fall (due to the decrease in income) by the same amount as the increase in \( \mu_0 \). (Although \( \mu_0 \) doesn’t show up in the solution for \( \tau_t \), it will have short-run effects on \( \tau_t \). But this is an artifact of how I set up the problem. To think about the transition to a new steady-state when \( \mu_0 \) changes at a point in time, we should define \( \mu_0 \) as \( \mu_0(t) \).)

d. You will notice that the four equations of the model entail aggregate demand behavior only. The exogenous price assumption essentially means that aggregate supply is perfectly elastic at the given price. Endogenizing the price level requires relaxing the assumption of perfectly elastic supply and including assumptions regarding the behavior and firms and households in input markets, and an assumption about technology that shows how inputs are used to produce output. For example, we could include labor supply (household) and labor demand (firm) decisions, and assume a production function relating the variable input labor to output. If we further assume that nominal wages are rigid and labor demand determines employment, the price level will be endogenous.

2. This assumption will affect the computation of \( E_{t-1}x_t \) in the reduced form for \( p_t \) (see p. 33 of the notes). Since \( E_{t-1}e^d_t \equiv e^d_t \). The reduced form for output will be:

\[
y_t = \gamma + \frac{a_1 b_1}{a_1 + b_1} (M_t - E_{t-1} M_t) + \left( \frac{b_1}{a_1 + b_1} \right) \epsilon^s_t.
\]

The demand shock drops out of the reduced form for output. Suppose such a shock is positive; then it will begin to drive up price as aggregate demand exceeds aggregate supply (see the second equation of the model). But by the first equation, the expected price changes one for one with actual price, so (given the form of the supply curve), there is no incentive for firms to alter output – the relative price has changed. This result might change if the aggregate supply curve is different, even if expectations are rational.

3. Cagan’s model.

a. Given the equation generating \( m \), it is easy to show that

\[
m_{t+i} = \gamma^i m_t + \gamma^i \epsilon_{t+i-1} + \ldots + \epsilon_{t+k}
\]

Since the error terms are all expected to be zero as of time \( t \), we have the desired result.

b. Use the solution on page 23 of the overheads:

\[
p_t = \frac{1}{1 - \alpha} \sum_{i=0}^{\infty} \beta^i E_t m_{t+i}
\]

Using the result from part (a) in (*) , we get the equilibrium, reduced form price that is implied by the theory:

\[
p_t = [1 + \alpha(\gamma - 1)]^{-1} m_t.
\]
c. Write the price as a function of the $\epsilon$’s:

$$p_t = [1 + \alpha(\gamma - 1)]^{-1}(1 - \alpha)^{-1}\epsilon_t$$

d. Multipliers:

$$\frac{\partial p_{t+i}}{\partial \epsilon_t} = \left[\frac{\alpha^i}{1 + \alpha(\gamma - 1)}\right]^{-1}$$

which is a positive number that is less than or equal to one given the restrictions on the parameters. An increase in $\epsilon_t$ has a persistent effect on $m$ in the future; from (*), it is clear that these future effects are built into the current price level. If $\gamma < 1$, the shock to $\epsilon_t$ has only a temporary effect on $m_t$, and the long-run effect on price and money are zero. But in the short-run, the multiplier will be less than one, so price responds less than one for one with money. This has an interesting implication. Look at the theoretical equilibrium condition (the first equation of the problem): if $p$ rises by less than $m$, then the left hand side increases. Since $\alpha$ is negative, the right hand side will increase only if there is expected future price is lower than the current price. That is, the temporary rise in money causes expected deflation! This is because there is no permanent change in the growth rate of money, and the level changes only temporarily. In effect, the price level overshoots its long-run value, so it must decline in the future.

Think further about this model when $\gamma < 1$. Suppose that the increase in $m$ is matched by a proportional increase in $p$. The left side of the equilibrium equation would not change, requiring the expected future price to equal the current price. But since the rise in money is not permanent, it will fall in the future from its temporarily higher level, so that the price must fall from its temporarily higher level as well. But since expectations are rational, this future decline in the price level must be predicted. This is therefore inconsistent with the proportional increase in $p_t$ and $E_t p_{t+1}$ assumed above. Thus, $p$ must rise less than proportionally to $m$.

e. If $\gamma = 1$, the shock has a permanent effect on the level of money; thus, rational agents simply raise prices and expected prices proportionately. The multiplier in this case is identically equal to one for all periods. This works here because the rise in money is permanent, and therefore $m$, $p$ and the expectations of $p$ can all rise immediately one for one, maintaining equilibrium in the money market.