1. Consider the model of optimal dynamic consumption, where income is taken to be exogenous.
   a. Suppose that a permanent tax is imposed on both interest and labor income at a fixed rate $\rho$. Write the households dynamic budget constraint to include the tax. Analyze the effects of the tax on the optimal consumption path using a graphical approach.
   b. Suppose that the representative household must borrow at a higher interest rate than the rate at which it can lend. Graphically show how this case can be analyzed, and predict the effects on the optimal consumption path.

2. Consider now the model of optimal consumption and labor supply. Analyze the effect of a tax on labor income at constant rate $\rho$ on current and future labor supply assuming that utility is log-linear (see p. 45 of the overheads).

3. Using the phase diagram developed in class, describe the transitory dynamics of capital and $q$, and the effects on their steady-state values, of the following:
   a. an increase in the rate of depreciation ($\delta$).
   b. an increase in aggregate output.
   c. the imposition of a permanent, proportional tax on investment.

4. As we’ve seen in class, the condition for the stock of capital to be optimal (the second first-order condition) defines a difference equation in the shadow value, $q_t$ (see p. 49 of the overheads). Solve this difference equation forward for $q_t$ and interpret the result.
Answers

1. Dynamic consumption behavior
   a. If interest and labor income are taxed at rate $\rho$, the intertemporal budget constraint becomes:

   $$A_{t+1} = [1 + r(1 - \rho)]A_t + (1 - \rho)y_t - c_t, \forall t$$

   By imposing the transversality condition, it is straightforward to see that the imposition of such a tax (i.e. changing $\rho$ from 0 to positive) will rotate the budget line counterclockwise, much like a decrease in the interest rate. (However, since the tax is imposed on income as well as interest, the rotation point need not be the original no-borrowing/no-lending point.) Future consumption will unambiguously fall, while the effect on current consumption is ambiguous.

   b. The budget line would be “kinked” at the no-borrowing/no-lending point, with the flatter portion above and the steeper portion below: the budget line would become concave to the origin. The only complication to analyzing this case is non-differentiability at the kink, so a corner solution could exist.

2. If the tax is imposed on both current and future income, then there is no effect on labor supply, either current or future. Consumption in each period will be less by a factor of $(1-\rho)$. This result would likely change if the tax were temporary.

3. Investment
   a. An increase in $\delta$ shifts the q-locus to the left ($\delta$ increases the user cost, so for a given $q$, capital must fall so that its marginal return rises). Since capital is initially fixed, $q$ falls immediately to the new saddlepath; $q$ will then rise and capital will fall over time. In the new steady-state, $q = 1$ and capital is lower than before.

   b. An increase in output increases the demand for capital and its return. The results are just the opposite of (a).

   c. A tax on investment would work like a subsidy, but in the opposite direction. The k-locus will shift upwards; $q$ will jump up to the new saddlepath, and capital will fall to a lower steady-state value. $q$ will rise to a higher level.

4. Solving forward yields

   $$q_t = \sum_{i=0}^{\infty} \left( \frac{1}{1 + r} \right)^{i+1} \left[ \pi(K_{t+1+i} - \delta) \right]$$

   Thus, $q$ is the discounted value of future (net) returns from capital. See Romer’s discussion in section 8.3.