Systems of Differential Equations

1 A Pair of Linear Differential Equations

We consider the system of two linear first-order differential equations:

\[
\begin{align*}
\dot{x}(t) &= a_{11}x(t) + a_{12}y(t) + \tilde{f}_1 \\
\dot{y}(t) &= a_{21}x(t) + a_{22}y(t) + \tilde{f}_2
\end{align*}
\]

where \(x(t)\) and \(y(t)\) are two dynamic variables, \(a_{ij}\) are given constants and \(\tilde{f}_i\) are constant forcing functions and they may take the form of a once-and-for-all exogenous shift from one constant level to the other. The solution to the above system will be two functions \(x(t)\) and \(y(t)\) that satisfy (1) and will consist of the general solution related to the homogeneous part of the system plus a particular solution related to the non-homogeneous part.

The homogeneous system corresponding to (1) is

\[
\begin{align*}
\dot{x} &= a_{11}x + a_{12}y \\
\dot{y} &= a_{21}x + a_{22}y
\end{align*}
\]

where the time component of the functions have been dropped for notational convenience.

There is more than one method of finding a solution to the homogeneous system (2). One way is to posit the following arbitrary solutions for \(x\) and \(y\):

\[
x(t) = Ae^{\mu t}, \quad y(t) = Be^{\mu t}
\]

Substitution of these proposed solutions into (2) yields:

\[
\begin{align*}
\mu Ae^{\mu t} &= a_{11}Ae^{\mu t} + a_{12}Be^{\mu t} \\
\mu Be^{\mu t} &= a_{21}Ae^{\mu t} + a_{22}Be^{\mu t}
\end{align*}
\]

Divide (4) throughout by \(e^{\mu t}\), collect terms, and rewrite in matrix notation:
\[
\begin{pmatrix}
  a_{11} - \mu & a_{12} \\
  a_{21} & a_{22} - \mu
\end{pmatrix}
\begin{pmatrix}
  A \\
  B
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0
\end{pmatrix}
\quad (5)
\]

A trivial solution to (5) is that \( A = B = 0 \), and is not of interest to us. If we must have \( A \neq 0 \) and \( B \neq 0 \), then it must be the case that the coefficient matrix in (5) is singular, i.e., its determinant must be zero:

\[
\begin{vmatrix}
  a_{11} - \mu & a_{12} \\
  a_{21} & a_{22} - \mu
\end{vmatrix}
= 0 
\quad (6)
\]

Expanding (6) yields a quadratic equation in \( \mu \):

\[
\mu^2 - (a_{11} + a_{22}) \mu + (a_{11}a_{22} - a_{21}a_{12}) = 0 
\quad (7)
\]

(7) is called the characteristic equation of (2). It has roots

\[
\mu = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{12})}}{2} 
\quad (8)
\]

(8) implies that the system (2) has two characteristic roots or eigenvalues associated with it. Let us denote these two roots by \( \mu_1 \) and \( \mu_2 \). Also note that these eigenvalues satisfy the following conditions:

\[
\mu_1 \mu_2 = a_{11}a_{22} - a_{12}a_{21} \quad \text{and} \quad \mu_1 + \mu_2 = a_{11} + a_{22} 
\quad (9)
\]

If \( \mu_1 \neq \mu_2 \), then the general solution to (2) is:

\[
x(t) = A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t} \\
y(t) = B_1 e^{\mu_1 t} + B_2 e^{\mu_2 t} 
\quad (10)
\]

where \( A_1 \) and \( A_2 \) are determined from initial conditions on \( x(t) \) and \( y(t) \), and \( B_1 \) and \( B_2 \) are determined from (4):

\[
B_1 = \frac{(\mu_1 - a_{11})}{a_{12}} A_1; \quad B_2 = \frac{(\mu_2 - a_{11})}{a_{12}} A_2 
\quad (11)
\]

We have now found the general solution to the homogeneous system (2). A particular solution for the non-homogeneous part of (1) can be found by assuming that \( x = \overline{x}, y = \overline{y} \), so that \( \dot{x} = \dot{y} = 0 \). Then a particular constant solution can be found by solving the system of equations:
\[ a_1x + a_{12}y + \tilde{f}_1 = 0 \]
\[ a_2x + a_{22}y + \tilde{f}_2 = 0 \]  \hspace{1cm} (12)

The solution \((x, y)\) to (12) where \(\dot{x} = \dot{y} = 0\) is called an equilibrium point, steady state, or stationary point. For our purposes, we will henceforth call it the steady-state equilibrium point.

Therefore, the general solution to (1) can then be written as:

\[ x(t) = \bar{x} + A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t} \]
\[ y(t) = \bar{y} + \frac{(\mu_1 - a_{11})}{a_{12}} A_1 e^{\mu_1 t} + \frac{(\mu_2 - a_{11})}{a_{12}} A_2 e^{\mu_2 t} \]  \hspace{1cm} (13)

2 Non-Linear Differential Equation Systems

Consider the following non-linear differential equation system:

\[ \dot{x} = f(x, y) \]
\[ \dot{y} = g(x, y) \]  \hspace{1cm} (14)

Suppose that this system has an isolated steady state equilibrium at \((\bar{x}, \bar{y})\), i.e., at \(x = \bar{x}\) and \(y = \bar{y}\),

\[ f(\bar{x}, \bar{y}) = 0 \]
\[ g(\bar{x}, \bar{y}) = 0 \]  \hspace{1cm} (15)

and there is a neighborhood of \((\bar{x}, \bar{y})\) containing no other equilibria of (14). In order to analyze the non-linear system (14), we first need to approximate it to a linear differential equation system. The approximation is done in the neighborhood of \((\bar{x}, \bar{y})\), by taking a first-order Taylor series expansion (retaining only linear terms) of the right-hand side of (14) around the steady state \((\bar{x}, \bar{y})\). Thus,

\[ \dot{x} = f(\bar{x}, \bar{y}) + f_x(\bar{x}, \bar{y})(x - \bar{x}) + f_y(\bar{x}, \bar{y})(y - \bar{y}) \]
\[ \dot{y} = g(\bar{x}, \bar{y}) + g_x(\bar{x}, \bar{y})(x - \bar{x}) + g_y(\bar{x}, \bar{y})(y - \bar{y}) \]  \hspace{1cm} (16)

However, taking into account (15), this reduces to

\[ \dot{x} = a_{11}(x - \bar{x}) + a_{12}(y - \bar{y}) \]
\[ \dot{y} = a_{21}(x - \bar{x}) + a_{22}(y - \bar{y}) \]  \hspace{1cm} (17)
where the constants are

\[ a_{11} = f_x(x, y); a_{12} = f_y(x, y); a_{21} = g_x(x, y); a_{22} = g_y(x, y) \] (18)

In the neighborhood of \((x, y)\), the solution to (14) behaves like that of the solution to (17), and the behavior of (17) can be determined following the analysis laid out in section 1.

From (17), we can express the **linearized dynamics** in the following form:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\
a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x - x_0 \\
y - y_0 \end{pmatrix}
\] (19)

The general solution to (19) will also be represented by (13).

### 3 Equilibrium Behavior

Recall that the solution to (1) or (14) consists of two characteristic roots or eigen values, \((\mu_1, \mu_2)\), such that

\[
\mu_1\mu_2 = a_{11}a_{22} - a_{12}a_{21} \quad \text{(determinant of coefficient matrix of (19))}
\]

and

\[
\mu_1 + \mu_2 = a_{11} + a_{22} \quad \text{(Trace of coefficient matrix of (19))}
\]

The behavior of the dynamic system (1) or (14) depends on the relationship between the two eigen values. We can identify several cases that describe the possible magnitudes and signs of \(\mu_1\) and \(\mu_2\). We will list three important cases here.

**CASE 1** *The eigen values \((\mu_1, \mu_2)\) are real and distinct, with \(\mu_1 < \mu_2 < 0\).*

The equilibrium is **stable** in this case. As a consequence, note that

\[
\mu_1\mu_2 = a_{11}a_{22} - a_{12}a_{21} > 0
\]

\[
\mu_1 + \mu_2 = a_{11} + a_{22} < 0
\]

**CASE 2** *The eigen values \((\mu_1, \mu_2)\) are real and distinct, with \(\mu_1 > \mu_2 > 0\).*

The equilibrium is **unstable** in this case. Since both roots are positive, the dynamic variables grow without bound with time. Note that

\[
\mu_1\mu_2 = a_{11}a_{22} - a_{12}a_{21} > 0 \quad \text{and} \quad \mu_1 + \mu_2 = a_{11} + a_{22} > 0
\]
CASE 3  The eigen values \((\mu_1, \mu_2)\) are real and distinct, with \(\mu_1 > 0 > \mu_2\).

If \(A_1 \neq 0\), then the term in (13) with the positive root will dominate the solution and both \(x\) and \(y\) will grow without bound. However, if \(A_1 = 0\) and \(A_2 \neq 0\), then the solution (13) will converge to the equilibrium \((\bar{x}, \bar{y})\), as \(t\) increases. This equilibrium is called a \textit{saddle-point}. Note that

\[
\mu_1 \mu_2 = a_{11}a_{22} - a_{12}a_{21} < 0 \quad \text{and the sign of } \mu_1 + \mu_2 = a_{11} + a_{22} \text{ is undetermined.}
\]