Saddle-Point Behavior and Phase Diagrams

Suppose we have the following non-linear system of differential equations

\begin{align*}
\dot{x} &= f(x, y) \\
\dot{y} &= g(x, y)
\end{align*}

(1)

Using a first-order Taylor series expansion, we can represent the above system (1) in the following matrix form

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
x - x^* \\
y - y^*
\end{pmatrix}
\]

(2)

where, \(a_{11} = f_x(x^*, y^*); a_{12} = f_y(x^*, y^*); a_{21} = g_x(x^*, y^*); a_{22} = g_y(x^*, y^*)\) and \((x^*, y^*)\) are the steady-state values of \(x\) and \(y\), respectively. Suppose that the solution to (1) consists of two eigenvalues \((\mu_1, \mu_2)\) such that \(\mu_1 < 0\) and \(\mu_2 > 0\). Therefore, the equilibrium is a saddle-point with one stable and one unstable eigenvalue.

In order to maintain stability, one of the variables, say \(y\), must be a “jump” variable (or a “control” variable) which can respond instantaneously to new information and/or shocks, while the other variable, \(x\), must be a “sluggish” variable (or a “state” variable) which is constrained to evolve continuously at all times. Therefore, unlike the “jump” variable \(y\), \(x\) cannot change instantaneously in response to new information.\(^1\)

Constructing the Phase Diagram

The phase diagram is a graphical representation of the dynamic behavior of the linearized system (2) in the close neighborhood of the steady-state equilibrium. The construction of the phase diagram involves two steps: (i) construction of the equilibrium loci, and (ii) deriving the “arrows of motion” in the close neighborhood of the equilibrium loci.

(i) The Equilibrium Loci

An equilibrium locus represents all combinations of the two variables \(x\) and \(y\) along which the steady state equilibrium conditions are maintained. Therefore, the equilibrium loci will be depicted in the \(y-x\) space. By convention, we shall plot the jump (control) variable \(y\) on the vertical axis and the sluggish (state) variable \(x\) on the horizontal axis. Since the dynamic system (1) is a second-order (two-variable) system, there will be two equilibrium loci, one for each dynamic variable.

\(^1\) In higher order systems, the number of jump (control) variables must be equal to the number of unstable eigenvalues.
(a) The $\dot{y} = 0$ locus

This locus represents combinations of $y$ and $x$ such that $\dot{y} = 0$. Therefore,

$$\dot{y} = g(x, y) = 0$$

(3)

Totally differentiating both sides of (3),

$$g_y \, d\dot{y} + g_x \, d\dot{x} = 0$$

$$\Rightarrow \quad \left( \frac{d\dot{y}}{d\dot{x}} \right)_{\dot{y}=0} = - \frac{g_x}{g_y} = - \frac{a_{21}}{a_{22}} .$$

(3a)

(3a) represents the slope of the $\dot{y} = 0$ locus. Since we do not have an explicitly specified model, let us assume, for the sake of exposition, that

$$\left( \frac{d\dot{y}}{d\dot{x}} \right)_{\dot{y}=0} < 0$$

Therefore, the $\dot{y} = 0$ locus is assumed to be downward sloping in the $y$-$x$ space.

(b) The $\dot{x} = 0$ locus

This locus represents combinations of $y$ and $x$ such that $\dot{x} = 0$. Therefore,

$$\dot{x} = f(x, y) = 0$$

(4)

Totally differentiating both sides of (4),

$$f_y \, d\dot{y} + f_x \, d\dot{x} = 0$$

$$\Rightarrow \quad \left( \frac{d\dot{y}}{d\dot{x}} \right)_{\dot{x}=0} = - \frac{f_x}{f_y} = - \frac{a_{11}}{a_{12}} .$$

(4a)

(4a) represents the slope of the $\dot{x} = 0$ locus. Since we do not have an explicitly specified model, let us assume, for the sake of exposition, that

$$\left( \frac{d\dot{y}}{d\dot{x}} \right)_{\dot{x}=0} > 0$$

Therefore, the $\dot{x} = 0$ locus is assumed to be upward sloping in the $y$-$x$ space.
(ii) The Arrows of Motion

The arrows of motion indicate the dynamic behavior of the system when a shock causes it to move away from its initial (pre-shock) equilibrium path (loci). In other words, the arrows of motion determine the **transitional dynamics** of the system, i.e., its dynamic behavior between the old and new steady states, following a shock.

(a) **Arrows of motion around the \( \dot{y} = 0 \) locus**

Since the jump variable \( y \) is represented on the vertical axis, the arrows of motion associated with the \( \dot{y} = 0 \) locus will also be in the vertical direction (‘up’ or ‘down’). The intuitive question one must ask here is as follows: *Given the \( \dot{y} = 0 \) locus, if \( x \) changes such that \( \dot{y} \neq 0 \) (i.e., the change in \( x \) takes the system off its equilibrium path), in which direction does \( y \) change?*

Since the system is off its equilibrium path, \( \dot{y} \neq 0 \). Therefore, at any point outside the \( \dot{y} = 0 \) locus, we have \( \dot{y} = g(x, y) \). Partially differentiating with respect to \( x \), we get

\[
\frac{\partial \dot{y}}{\partial x} = g_x = a_{21}
\]

Again, for the sake of exposition, let us assume that \( a_{21} > 0 \). This implies that as \( x \) increases, the change in \( \dot{y} \) is positive, i.e., \( y \) increases over time. Hence, at any point **above** or **to the right** of the \( \dot{y} = 0 \) locus, \( y \) is increasing and at any point **below** or **to the left** of the \( \dot{y} = 0 \) locus, \( y \) is decreasing (in the vertical direction). This is depicted in the figure below:

![Diagram](attachment://figure1.png)

Figure 1
(b) **Arrows of motion around the \( \dot{x} = 0 \) locus**

Since the sluggish variable \( x \) is represented on the horizontal axis, the arrows of motion associated with the \( \dot{x} = 0 \) locus will also be in the horizontal direction (‘left’ or ‘right’). The intuitive question one must ask here is as follows: *Given the \( \dot{x} = 0 \) locus, if \( y \) changes such that \( \dot{x} \neq 0 \) (i.e., the change in \( y \) takes the system off its equilibrium path), in which direction does \( x \) change?*

Since the system is off its equilibrium path, \( \dot{x} \neq 0 \). Therefore, at any point outside the \( \dot{x} = 0 \) locus, we have \( \dot{x} = f(x, y) \). Partially differentiating with respect to \( y \), we get

\[
\frac{\partial \dot{x}}{\partial y} = f_y = a_{12}
\]

Again, for the sake of exposition, let us assume that \( a_{12} > 0 \). This implies that as \( y \) increases, the change in \( \dot{x} \) is positive, i.e., \( x \) increases over time. Hence, at any point **above** or **to the left of** the \( \dot{x} = 0 \) locus, \( x \) is increasing and at any point **below** or **to the right of** the \( \dot{x} = 0 \) locus, \( x \) is decreasing (in the horizontal direction). This is depicted in the figure below:

![Figure 2](image-url)
Combining figures 1 and 2 completes the construction of the phase diagram:

![Phase Diagram]

The point $S$ at which the two equilibrium loci $\dot{x} = 0$ and $\dot{y} = 0$ intersect is the steady-state equilibrium point, also called the “saddle-point”. The arrows in figure 3 represent the resultant directions of motion of the dynamic system when it is off the equilibrium loci. The arrows of motion indicate that there is a unique path, $X'X'$, called the **saddle-path** which leads to the saddle-point $S$. Given the initial value of the sluggish variable $x(0)$, at time $t = 0$, $y$ jumps instantaneously on to the saddle-path $X'X'$, following which the system converges to its steady-state equilibrium point $S$ along $X'X'$.

**Saddle-point Behavior**

Let us now modify the dynamic system (1) to include a “forcing” function or shock. The modified system can be written as

\[
\begin{align*}
\dot{x} &= f(x, y) + \bar{e}_1 \\
\dot{y} &= g(x, y) + \bar{e}_2
\end{align*}
\]

(5)

where $\bar{e}_i (i = 1, 2)$ represent constant values of some forcing function, and can be interpreted as exogenous shocks to the system. Let the corresponding initial (pre-shock) steady-state equilibrium solutions for $x$ and $y$ be denoted by $\bar{x}_1$ and $\bar{y}_1$, respectively. Suppose at time $t = 0$ it is announced that $\bar{e}_1$ and $\bar{e}_2$ are to increase at time $T \geq 0$, to $\bar{e}_1$ and $\bar{e}_2$, respectively. Let the new (after-shock) steady-state equilibrium solutions be denoted by $\bar{x}_2$ and $\bar{y}_2$. 
Before the above shock occurs, over the period $0 < t < T$, the solutions to $x$ and $y$ are of the form

$$x = \bar{x}_1 + A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t}$$

$$y = \bar{y}_1 + \left( \frac{\mu_1 - a_{11}}{a_{12}} \right) A_1 e^{\mu_1 t} + \left( \frac{\mu_2 - a_{11}}{a_{12}} \right) A_2 e^{\mu_2 t}$$

(6a)

(6b)

Note that since $\mu_i$ are eigen values,

$$\frac{\mu_i - a_{11}}{a_{12}} = \frac{a_{21}}{\mu_i - a_{22}}, \quad i = 1, 2.$$  

After the shock has occurred, over the period $t \geq T$, the corresponding solutions for $x$ and $y$ are

$$x = \bar{x}_2 + A_1' e^{\mu_1' t} + A_2' e^{\mu_2' t}$$

$$y = \bar{y}_2 + \left( \frac{\mu_1 - a_{11}}{a_{12}} \right) A_1' e^{\mu_1' t} + \left( \frac{\mu_2 - a_{11}}{a_{12}} \right) A_2' e^{\mu_2' t}$$

(7a)

(7b)

In order to complete the solution, we need to determine the arbitrary constants $A_i$ and $A'_i (i = 1, 2)$. To do this, we will impose the following conditions:

**Condition 1: The solution must be bounded**

In order for $x$ and $y$ not to diverge as $t \to \infty$, we must have $A_2' = 0$ so that

$$x = \bar{x}_2 + A_1' e^{\mu_1' t}$$

$$y = \bar{y}_2 + \left( \frac{\mu_1 - a_{11}}{a_{12}} \right) A_1' e^{\mu_1' t}$$

(8a)

(8b)

i.e., after $t \geq T$, $x$ and $y$ must follow the stable paths described by (8a) and (8b). Eliminating $A'_1 e^{\mu_1' t}$ from (8a) and (8b) yields

$$y - \bar{y}_2 = \left( \frac{\mu_1 - a_{11}}{a_{12}} \right) (x - \bar{x}_2)$$

(9)

This is a locus in the $y$-$x$ space that describes the stable-arm (saddle-path) of the saddle-point equilibrium.
Condition 2: *Impose initial conditions*

Let \( x(0) = \bar{x}_1 \), the initial, pre-shock steady-state equilibrium. Imposing this condition, we can re-write (6a) as

\[
A_1 + A_2 = 0 \tag{10}
\]

Condition 3: *The solution must be continuous at \( t = T \)*

This condition implies that the solutions for \( x \) and \( y \) given in (6a), (6b), (8a) and (8b) must coincide at \( t = T \), thereby precluding any jumps that are foreseen. Therefore, we must have

\[
\bar{x}_1 + A_1 e^{\mu T} + A_2 e^{\mu_2 T} = \bar{x}_2 + A'_1 e^{\mu_1 T} \tag{11a}
\]

\[
\bar{y}_1 + \left( \frac{\mu_1 - a_{11}}{a_{12}} \right) A_1 e^{\mu T} + \left( \frac{\mu_2 - a_{11}}{a_{12}} \right) A_2 e^{\mu_2 T} = \bar{y}_2 + \left( \frac{\mu_1 - a_{11}}{a_{12}} \right) A'_1 e^{\mu_1 T} \tag{11b}
\]

Re-arranging (11a) and (11b), we have

\[
(A_1 - A'_1)e^{\mu T} + A_2 e^{\mu_2 T} = \bar{x}_2 - \bar{x}_1 = dx \tag{12a}
\]

\[
\left( \frac{\mu_1 - a_{11}}{a_{12}} \right) (A_1 - A'_1)e^{\mu T} + \left( \frac{\mu_2 - a_{11}}{a_{12}} \right) A_2 e^{\mu_2 T} = \bar{y}_2 - \bar{y}_1 = d\bar{y} \tag{12b}
\]

Substituting (12a) and (12b) into (6a), (6b) and (8a), (8b) yields the solutions for \( A_i \) and \( A'_i \) \( (i = 1, 2) \).

In summary, the dynamics involve three phases:

(I) Announcement at time \( t = 0 \), of a shift to occur at time \( t = T \geq 0 \) in the future generates an immediate response in the jump variable. Setting \( t = 0 \) in (6b) and using (10), the initial response can be calculated as

\[
y(0) - \bar{y}_1 = \left( \frac{\mu_2 - \mu_1}{a_{12}} \right) A_2
\]

Also, the size of the initial jump is inversely proportional to the lead time, \( T \).

(II) After the initial jump, the system follows the unstable locus (7a) and (7b) until time \( t = T \), when the announced policy is implemented.
(III) At $t = T$, the system reaches the stable locus (9), which it then follows until it reaches the new equilibrium.

Note: If $T = 0$, and the shift is unannounced, the system jumps instantaneously to the new stable locus (9).