The Hamiltonian for the private agent is:

\[
H = \frac{1}{\gamma} (C_T C_N^\theta)^{\gamma - \rho b} + \lambda e^{\rho b} \left[ n b + Y_T - C_T + p \{ Y_N - C_N - I \} - T_L - \delta \right]
\]

Optimality conditions: (Note that \( k_i = K_i / L_i \); \( i = N, T \))

\[
\begin{align*}
C_T^{-\lambda} C_N^{\theta} &= \lambda \\
\Theta C_T^{\gamma} C_N^{\theta - 1} &= \lambda p \\
\rho - \frac{\lambda}{\lambda} &= r
\end{align*}
\]

\[
\begin{align*}
\alpha A_T k_T^{\alpha - 1} k_g^{\eta(1 - \alpha)} &= \beta A_N k_N^{\beta - 1} k_g^{\eta(1 - \beta)} \\
(1 - \alpha) A_T k_T^{\alpha - 1} k_g^{\eta(1 - \alpha)} &= \beta (1 - \beta) A_N k_N^{\beta - 1} k_g^{\eta(1 - \beta)} \\
\beta A_N k_N^{\beta - 1} k_g^{\eta(1 - \beta)} + \frac{\delta}{\beta} &= r
\end{align*}
\]

[Note: (1f) is derived noting that \( \dot{k} = I \) and \( \partial H / \partial k_T = \partial H / \partial k_N = \partial H / \partial k \)]

Finally, the usual transversality conditions apply for \( b \) and \( k \).

Eq. (1a) and (1b) equate the MU of consumption from the traded and non-traded goods to the MU of wealth. From (1c), given exogenous \( \rho \) and \( r \) (world interest rate), we require that \( \dot{\lambda} = 0 + \delta \), which implies a constant MU of wealth, i.e., \( \lambda = \bar{\lambda} + \delta t \). Eqs. (1d) and (1e) equate the marginal returns from allocating capital and labor across the two sectors. Eq. (1f) represents a no-arbitrage condition for capital (non-traded). Since it is only non-traded capital that can be converted to investment, the total return to capital consists of the marginal product of non-traded capital plus any capital gains/losses from fluctuations in its relative price, \( \rho \).
From (1a) and (1b) we get
\[ C_N = \frac{B}{\beta} C_T \]

Using the above in (1a) we can solve for \( C_T \):
\[ C_T = \left[ \frac{\lambda (f \beta)^{\frac{1}{\theta}}}{1 - \theta} \right]^\frac{1}{\theta (1 - \theta) - 1} \quad (2a) \]

Similarly,
\[ C_N = \left[ \frac{\lambda (f \beta)^{1 - \gamma}}{1 - \theta} \right]^\frac{1}{\theta (1 - \theta) - 1} \quad (2b) \]

Dividing (1d) by (1e) we get
\[ k_N = \left[ \frac{1 - \alpha}{1 - \rho} \cdot \frac{\beta}{\alpha} \right] k_T \quad (3) \]

Using (3) in (1d), we can solve for \( k_T \):
\[ k_T = \Delta_1 \beta^{\frac{1}{\alpha} - \beta} \cdot K^2 \quad (4a) \]

where
\[ \Delta_1 = \left[ \left( \frac{1 - \alpha}{1 - \rho} \cdot \frac{\beta}{\alpha} \right)^{\frac{1}{\alpha} - \beta} \left( \frac{\beta A_N}{\alpha A_r} \right) \right]^{\frac{1}{\alpha} - \beta} \]

It immediately follows, from (3) that
\[ k_N = \Delta_2 \beta^{\frac{1}{\alpha} - \beta} K^2 \quad (4b) \]

where
\[ \Delta_2 = \left[ \left( \frac{1 - \alpha}{1 - \rho} \cdot \frac{\beta}{\alpha} \right)^{\frac{1}{\alpha} - \beta} \left( \frac{\beta A_N}{\alpha A_r} \right) \right]^{\frac{1}{\alpha} - \beta} \]

Note that
\[ K = k_T + k_N \quad (5) \]
\[ \Rightarrow \quad K = k_T L_T + k_N L_N \quad (5') \]

Since \( L_T + L_N = 1 \), we can write (5') as
\[ K = k_T L_T + k_N (1 - L_T) \quad (5'') \]
Solving (5") for $L_T$:

$$L_T = \frac{K - k_n}{k_T - k_n}$$

Using (4a) and (4b),

$$L_T = L_T(p, K, K_g) = \frac{K - k_n(p, K_g)}{k_T(p, K_g) - k_n(p, K_g)}$$

Eqs. (4a), (4b) and (6) enable us to characterize $Y_N$ and $Y_T$:

$$Y_N = A_T k_T^\beta K_g^{\eta(-\beta)} (1 - L_T) = Y_N(p, K, K_g)$$

$$Y_T = A_T k_T^\alpha K_g^{\eta(-\alpha)} L_T = Y_T(p, K, K_g)$$

**Equilibrium Dynamics:**

There are 4 dynamic equations that describe the evolution of the economy. First, we have the goods market equilibrium condition in the non-traded sector:

$$Y_N = C_N + I_N = C_N + K$$

$$\Rightarrow K = Y_N(p, K, K_g) - C_N(\lambda, p)$$

where $Y_N(\cdot)$ and $C_N(\cdot)$ are given by (7a) and (2b), respectively.

Second, we have the no-arbitrage condition for capital (non-traded) from (4f):

$$p = p[p - \rho A_T k_T^{\beta-1} K_g^{\eta(-\beta)}]$$

Third, the evolution of public capital is given by:

$$k_g = g[Y_T + p Y_N] - \delta_g K_g$$

Finally, the current account dynamics are given by:

$$b = Y_T - C_T + rb$$
Of the 4 equations (8a) - (8d), only 3 are independent (\( \tilde{\epsilon}, \tilde{\kappa}, \tilde{\nu} \)) while the evolution of the current account depends on the solution to (8a) - (8c). Therefore, the core dynamics are characterized by:

\[
\begin{align}
\dot{K} &= Y_n(p, K, \tilde{K}_g) - \nu_n(\tilde{\lambda}, \tilde{\nu}) \quad (9a) \\
\dot{\tilde{K}} &= g \left[ Y_n(p, K, \tilde{K}_g) + \frac{1}{\beta} \nu_n(p, K, \tilde{K}_g) \right] - \delta_g \tilde{K}_g \quad (9b) \\
\dot{\tilde{\nu}} &= \frac{1}{\beta} \left[ \nu_A K_n^{\beta-1} \tilde{K}_g^{\gamma(1-\beta)} \right] \quad (9c)
\end{align}
\]

The linearized dynamic system can be represented by:

\[
\dot{\tilde{X}} = [J] (\tilde{X} - \tilde{X}')
\]

where \( \tilde{X}' = (K, \tilde{K}, \tilde{\nu}) \) and \( \tilde{X}' = (\tilde{K}, \tilde{\nu}, \tilde{\kappa}) \) and \([J]\) is a 3x3 matrix of coefficients \( a_{ij} \) \((i, j = 1, 2, 3)\). It can be demonstrated that the equilibrium is a saddle path, with \( K \) and \( \tilde{K}_g \) being the sluggish adjustment variables and \( \tilde{\nu} \) being the "jump" variable. The system will have 2 stable (negative) and 1 unstable (positive) eigenvalue.

**Steady-State**

The steady-state is attained when \( k = \tilde{k} = \tilde{\nu} = 0 \) so that:

\[
\begin{align}
A_A K_n^{\beta} \tilde{K}_g^{\gamma(1-\beta)} &\left[ 1 - L_T(\tilde{\kappa}, \tilde{K}_n, \tilde{K}_g) \right] = \left[ \tilde{A}_1 (\tilde{\kappa}/\theta)^{1-\gamma} \right]^{1/(\gamma+\delta)} \quad (10a) \\
g \left[ A_t K_t^{\gamma(1-\beta)} L_T(\tilde{\kappa}, \tilde{K}_n, \tilde{K}_g) + \beta A_K K_n^{\gamma(1-\beta)} \right] &\left[ 1 - L_T(\tilde{\kappa}, \tilde{K}_n, \tilde{K}_g) \right] = \delta_g \tilde{K}_g \quad (10b) \\
\tilde{\kappa} &= \beta A_K K_n^{\beta-1} \tilde{K}_g^{\gamma(1-\beta)} \quad (10c)
\end{align}
\]

Also, the following applies in the steady state:

\[
\begin{align}
\tilde{K} &= L_T(\tilde{\kappa}, \tilde{K}_n, \tilde{K}_g) K_t + \left[ 1 - L_T(\tilde{\kappa}, \tilde{K}_n, \tilde{K}_g) \right] K_n \quad (11a) \\
\dot{\tilde{\kappa}} &= g \left[ A_t K_t^{\gamma(1-\beta)} L_T(\tilde{\kappa}, \tilde{K}_n, \tilde{K}_g) - \left[ \tilde{A}_1 (\tilde{\kappa}/\theta)^{1-\gamma} \right]^{1/(\gamma+\delta)} + r \tilde{\kappa} = 0 \quad (11b) \\

\end{align}
\]

We therefore are left with one equation to describe the dynamics.
Given that \( k_r = k_r(p, K_g) \) and \( k_n = k_n(p, K_g) \), eqns. (11a) - (11e) can be solved for \( \{\tilde{K}, \tilde{K}_g, \tilde{b}, \tilde{p}, \tilde{\lambda}\} \). Once these are known, the sectoral allocations, \( \{\tilde{L}_r, \tilde{L}_n\} \) and \( \{\tilde{L}_T, \tilde{L}_N\} \) are immediately known.

Getting back to (9c):

\[
\bar{p} = p \left[ r - \beta A_N k_n^{\beta-1} K_g^{(1-\beta)} \right]
\]

where from (46) we see that:

\[
k_n = \Delta_2 p^{1/\lambda - p} K_g^p
\]

\[
\Rightarrow \quad \bar{p} = p \left[ r - \beta A_N \Delta_2 \left[ \frac{p}{1 - p} K_g^p \right]^{\beta-1} K_g^{(1-\beta)} \right]
\]

\[
\Rightarrow \quad \bar{p} = p \left[ r - \beta A_N \left[ \Delta_2 p^{1/\lambda - p} \right]^{\beta-1} K_g^{(1-\beta)} \right]
\]

\[
\Rightarrow \quad \bar{p} = p \left[ r - \beta A_N \left[ \Delta_2 p^{1/\lambda - p} \right]^{\beta-1} K_g^{(1-\beta)} \right] \quad (12)
\]

It is clear from (12) that \( \bar{p} \) is independent of \( K_g \). Consequently, \( \bar{p} \) will also be independent of \( K_g \). Therefore, government spending shocks, unlike of their productive nature, do not affect the real exchange rate (in transition or in the long-run).

The steady-state real exchange rate can be obtained from (12) by setting \( \bar{p} = 0 \):

\[
\bar{p} = \left[ \frac{\beta A_N}{r \Delta_2^{1/\lambda - p}} \right]^{\alpha - \beta/1 - p}
\]

The real exchange rate is determined by supply-side variables only. Therefore, these results suggest that the standard two-sector dependent economy model’s predictions are robust to modifications about the nature of demand shocks.